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Page 239: on the left of the second formula, for ${}^{k+1}\xi^{p,\kappa}_{abc\ldots}$ read ${}^{k}\xi^{p,\kappa}_{abc\ldots}$

ELECTRICAL NOTES

By F. B. PIDDUCK (*Oxford*)

V. THE COMPTON EFFECT OF A BOUND ELECTRON

[Received 18 August 1931]

THE wave theory of the Compton effect of a free electron contains a dilemma which rather discounts the ease with which one can calculate the disturbed wave functions. The electron being unlocalized, radiation and matter are superposed everywhere, and the resulting theory,† though elegant, is hardly elementary. Gordon's theory,‡ extended to Dirac electrons by Klein and Nishina,§ limits the 'space available for the electron' arbitrarily so as to isolate the radiation at great distances. It is more satisfactory to treat of electrons bound within a small space by nuclear attraction. This was first done in terms of Schrödinger's wave function by Wentzel.|| An attempt to work with Dirac's wave functions led me to think that Wentzel's theory must be defective. Wentzel expressed the electric moment in the usual way by series with denominators $E_n - E_s + h\nu$, where E_n is the energy of the bound or recoil electron, ν the frequency of the incident light and s a variable of summation; and replaced each denominator by $h\nu$ when the energy of both bound and recoil electrons is small compared with $h\nu$. But, since the functions expanded contain exponentials whose index is of order ν, a finite part of the sum comes from terms for which E_s is comparable with $h\nu$. It seemed best in these circumstances to see what can be done without a series expansion.

Let a plane polarized wave fall on an atom whose centre is O in a direction $(\sigma, 0, \kappa)$, where $\sigma^2+\kappa^2 = 1$, the vector potential being

$$A = \left(-\frac{c\kappa P}{2\pi\nu}, -\frac{cQ}{2\pi\nu}, \frac{c\sigma P}{2\pi\nu}\right)\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right), \qquad (1)$$

so that

$$E_x : E_y : E_z = \kappa P : Q : -\sigma P, \qquad H_x : H_y : H_z = \kappa Q : -P : -\sigma Q. \qquad (2)$$

† E. Schrödinger, *Ann. d. Phys.* 82 (1927), 257.

‡ W. Gordon, *Zeitschr. f. Phys.* 40 (1927), 117.

§ O. Klein and Y. Nishina, *Zeitschr. f. Phys.* 52 (1928), 853.

|| G. Wentzel, *Zeitschr. f. Phys.* 43 (1927), 1; cf. also I. Waller, *Phil. Mag.* 4 (1927), 1228.

Let V be the potential of the nucleus and $-e$ the charge on the electron. Then

$$\left(-\frac{h}{2\pi i}\frac{\partial}{c\partial t}+\frac{eV}{c}+mc\right)\psi_1+\left[\frac{h}{2\pi i}\frac{\partial}{\partial z}+\frac{e\sigma P}{2\pi\nu}\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_3+$$
$$+\left[\frac{h}{2\pi i}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)+\frac{e}{2\pi\nu}(-\kappa P+iQ)\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_4=0,$$

$$\left(-\frac{h}{2\pi i}\frac{\partial}{c\partial t}+\frac{eV}{c}+mc\right)\psi_2+$$
$$+\left[\frac{h}{2\pi i}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)+\frac{e}{2\pi\nu}(-\kappa P-iQ)\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_3-$$
$$-\left[\frac{h}{2\pi i}\frac{\partial}{\partial z}+\frac{e\sigma P}{2\pi\nu}\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_4=0,$$

$$\left[\frac{h}{2\pi i}\frac{\partial}{\partial z}+\frac{e\sigma P}{2\pi\nu}\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_1+\left[\frac{h}{2\pi i}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)+\right.$$
$$\left.+\frac{e}{2\pi\nu}(-\kappa P+iQ)\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_2+$$
$$+\left(-\frac{h}{2\pi i}\frac{\partial}{c\partial t}+\frac{eV}{c}-mc\right)\psi_3=0,$$

$$\left[\frac{h}{2\pi i}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)+\frac{e}{2\pi\nu}(-\kappa P-iQ)\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_1-$$
$$-\left[\frac{h}{2\pi i}\frac{\partial}{\partial z}+\frac{e\sigma P}{2\pi\nu}\sin 2\pi\nu\left(t-\frac{\sigma x+\kappa z}{c}\right)\right]\psi_2+$$
$$+\left(-\frac{h}{2\pi i}\frac{\partial}{c\partial t}+\frac{eV}{c}-mc\right)\psi_4=0.$$

Write $$\epsilon=ec/4\pi\nu^2 h, \tag{3}$$

and denote the set of four wave functions of an undisturbed state by $\psi(n)\exp[-2\pi i\nu(n)t]$, where n is a symbol for the quantum numbers or parameters required to specify the state. Then if ϵP and ϵQ are small, a slightly disturbed wave function is

$$\psi=\psi(n)\exp[-2\pi i\nu(n)t]+\delta\psi^+\exp\left[-2\pi i\nu(n)t-2\pi i\nu t+2\pi i\nu\left(\frac{\sigma x+\kappa z}{c}\right)\right]+$$
$$+\delta\psi^-\exp\left[-2\pi i\nu(n)t+2\pi i\nu t-2\pi i\nu\left(\frac{\sigma x+\kappa z}{c}\right)\right],$$

where $\delta\psi^\pm$ are two functions of x, y, z given by the equations

$$\left.\begin{aligned}
&\left[1\pm\frac{h\nu(n)+eV+mc^2}{h\nu}\right]\delta\psi_1^\pm+\\
&\qquad+\left[\kappa\pm\frac{c}{2\pi i\nu}\frac{\partial}{\partial z}\right]\delta\psi_3^\pm+\left[\sigma\pm\frac{c}{2\pi i\nu}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\delta\psi_4^\pm=f_1\\
&\left[1\pm\frac{h\nu(n)+eV+mc^2}{h\nu}\right]\delta\psi_2^\pm+\\
&\qquad+\left[\sigma\pm\frac{c}{2\pi i\nu}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\delta\psi_3^\pm-\left[\kappa\pm\frac{c}{2\pi i\nu}\frac{\partial}{\partial z}\right]\delta\psi_4^\pm=f_2\\
&\left[\kappa\pm\frac{c}{2\pi i\nu}\frac{\partial}{\partial z}\right]\delta\psi_1^\pm+\left[\sigma\pm\frac{c}{2\pi i\nu}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\delta\psi_2^\pm+\\
&\qquad+\left[1\pm\frac{h\nu(n)+eV-mc^2}{h\nu}\right]\delta\psi_3^\pm=f_3\\
&\left[\sigma\pm\frac{c}{2\pi i\nu}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\delta\psi_1^\pm-\left[\kappa\pm\frac{c}{2\pi i\nu}\frac{\partial}{\partial z}\right]\delta\psi_2^\pm+\\
&\qquad+\left[1\pm\frac{h\nu(n)+eV-mc^2}{h\nu}\right]\delta\psi_4^\pm=f_4
\end{aligned}\right\}\quad(4)$$

and

$$\begin{aligned}f=\epsilon[&-i\sigma P\psi_3(n)+(i\kappa P+Q)\psi_4(n),-(-i\kappa P+Q)\psi_3(n)+i\sigma P\psi_4(n),\\&-i\sigma P\psi_1(n)+(i\kappa P+Q)\psi_2(n),-(-i\kappa P+Q)\psi_1(n)+i\sigma P\psi_2(n)].\end{aligned}\quad(5)$$

Let the wave function ψ correspond to the bound electron and a second function ψ' to the recoil electron. These two give rise to a volume and current density proportional to

$$\left.\begin{aligned}
&\rho(n,n')=-\psi_1\psi_1'^*-\psi_2\psi_2'^*-\psi_3\psi_3'^*-\psi_4\psi_4'^*,\\
&j_{x+iy}(n,n')=2(\psi_4\psi_1'^*+\psi_2\psi_3'^*),\quad j_{x-iy}(n,n')=2(\psi_1\psi_4'^*+\psi_3\psi_2'^*),\\
&j_z(n,n')=\psi_1\psi_3'^*-\psi_2\psi_4'^*+\psi_3\psi_1'^*-\psi_4\psi_2'^*.
\end{aligned}\right\}\quad(6)$$

Omitting the undisturbed term and that corresponding to radiation harder than the incident radiation, a typical term in (6) is

$$\psi_r\psi_s'^*=[\psi_r(n)\delta\psi_s'^{-*}+\delta\psi_r^+\psi_s(n')^*]\exp\left(-2\pi i\nu't+2\pi i\nu\frac{\sigma x+\kappa z}{c}\right),\quad(7)$$

where $$\nu'=\nu-\nu(n')+\nu(n).\quad(8)$$

Thus we only use the terms $\delta\psi^+$ and $\delta\psi'^{-*}$, solving equations (4) with the positive sign for the bound electron and with the negative sign for the recoil electron.

Let the electron be in the field of a nucleus Ze, and write

$$\gamma=2\pi Ze^2/hc,\qquad b=4\pi^2mZe^2/h^2.\quad(9)$$

Then if γ^2 is neglected, the normal state makes $h\nu(n)=mc^2$ and

$$\left.\begin{aligned} \psi_1(n) &= \tfrac{1}{2}i\gamma\exp(-br)[-A(1-\mu^2)^{\frac{1}{2}}\exp(-i\phi)+B\mu], \\ \psi_2(n) &= \tfrac{1}{2}i\gamma\exp(-br)[A\mu+B(1-\mu^2)^{\frac{1}{2}}\exp i\phi], \\ \psi_3(n) &= -B\exp(-br), \qquad \psi_4(n) = A\exp(-br), \end{aligned}\right\} \tag{10}$$

where A and B are complex constants, of equal modulus when the wave functions are normalized. We consider only those recoil electrons which have a given kinetic energy $(\beta-1)mc^2$, so that $h\nu(n')=\beta mc^2$. Their wave functions, which need not be normalized for our present purpose, are

$$\left.\begin{aligned} \psi_1(n')^* &= -i(j-l)F_j^*P_j^l\exp(-il\phi), \\ \psi_2(n')^* &= -iF_j^*P_j^{l+1}\exp[-i(l+1)\phi], \\ \psi_3(n')^* &= -(j+l)G_j^*P_{j-1}^l\exp(-il\phi), \\ \psi_4(n')^* &= G_j^*P_{j-1}^{l+1}\exp[-i(l+1)\phi]. \end{aligned}\right\} \tag{11}$$

Write $$\delta=\beta\gamma(\beta^2-1)^{-\frac{1}{2}}, \qquad k=2\pi mc(\beta^2-1)^{\frac{1}{2}}h^{-1}. \tag{12}$$

Then the radial functions F_j^*, G_j^* contain confluent hypergeometric functions of $2ikr$, and their asymptotic values at great distances are

$$F_j^*=i\left(\frac{\beta-1}{\beta+1}\right)^{\frac{1}{2}}r^{-i\delta-1}\exp(-ikr), \qquad G_j^*=r^{-i\delta-1}\exp(-ikr). \tag{13}$$

Assume with Wentzel that the energy of binding is small compared with the kinetic energy of the recoil electron, which is itself small compared with $h\nu$. Thus, if

$$\alpha=h\nu/mc^2, \tag{14}$$

γ^2 is small compared with $\beta-1$, and $\beta-1$ with α. The corpuscular theory of the Compton effect makes

$$\nu'=\frac{\nu}{1+\alpha(1-\cos\theta)}, \qquad \beta-1=\frac{\alpha^2(1-\cos\theta)}{1+\alpha(1-\cos\theta)}$$

for scattering at an angle θ, so that α must be small. The conditions, as Wentzel observed, justify the use of the asymptotic values (13) and are satisfied by medium Röntgen rays; thus when the K rays of molybdenum are scattered at 90° by hydrogen, $\gamma^2=5{\cdot}3\times10^{-5}$, $\beta-1=1{\cdot}2\times10^{-3}$ and $\alpha=3{\cdot}4\times10^{-2}$. We shall solve equations (4), when α is small, by first neglecting the term eV. This is reasonable since the potential at a distance of 10^{-9} cm. from a hydrogen nucleus is only 140 volts. The approximation so found will be shown to be a true one; i.e. we are not committing any discontinuous error like the change from a bound to a free electron. The attraction of the nucleus has, so to speak, exhausted itself in quantizing the system,

and produces only a small further effect in the equations of perturbation.

Considering first the recoil electron, the asymptotic value of $(c/2\pi i\nu)\partial\psi/\partial z$, where ψ contains an exponential of the form (13), is of the order of $(c/2\pi i\nu)\partial\psi/\partial r$ or of $ck\psi/2\pi\nu = (\beta^2-1)^{\frac{1}{2}}\psi/\alpha$, and the coefficient is finite when $\alpha \to 0$. Hence the limiting form of the first two of equations (4) is

$$(\pm 2/\alpha)\delta\psi_1 + (\text{finite})\delta\psi_3 + (\text{finite})\delta\psi_4 = f_1,$$
$$(\pm 2/\alpha)\delta\psi_2 + (\text{finite})\delta\psi_3 + (\text{finite})\delta\psi_4 = f_2,$$

and $f_1{:}f_2{:}f_3{:}f_4$ are of the order $\alpha^{-1}{:}\alpha^{-1}{:}1{:}1$. Hence the first two equations as simplified determine $\delta\psi_1$ and $\delta\psi_2$, and the last two then give $\delta\psi_3$ and $\delta\psi_4$, thus:

$$\left.\begin{aligned} &\delta\psi_1^{\pm} = \pm\tfrac{1}{2}\alpha f_1, \qquad \delta\psi_2^{\pm} = \pm\tfrac{1}{2}\alpha f_2, \\ &\delta\psi_3^{\pm} = f_3 - \left[\kappa \pm \frac{c}{2\pi i\nu}\frac{\partial}{\partial z}\right](\pm\tfrac{1}{2}\alpha f_1) - \left[\sigma \pm \frac{c}{2\pi i\nu}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right](\pm\tfrac{1}{2}\alpha f_2), \\ &\delta\psi_4^{\pm} = f_4 - \left[\sigma \pm \frac{c}{2\pi i\nu}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right](\pm\tfrac{1}{2}\alpha f_1) + \left[\kappa \pm \frac{c}{2\pi i\nu}\frac{\partial}{\partial z}\right](\pm\tfrac{1}{2}\alpha f_2). \end{aligned}\right\} \quad (15)$$

All the components of $\delta\psi$ are of the same order αf_1. Similar but simpler considerations apply to the bound electron, since b is small compared with k by Wentzel's hypothesis, and we need not keep the differential terms in (15).

To calculate ρ and j, write

$$rs \equiv \psi_r(n)\psi_s(n')^*. \quad (16)$$

Then since γ is small compared with $(\beta^2-1)^{\frac{1}{2}}$ and with α, symbols $1s$ and $2s$ can be neglected, and also products of α with $r1$ and $r2$. We find

$$\left.\begin{aligned} j_x(n,n') &= \alpha\epsilon(i\kappa P)(33+44)\exp[-2\pi i\nu' t + 2\pi i\nu(\sigma x + \kappa z)/c] \\ j_y(n,n') &= \alpha\epsilon(iQ)(33+44)\exp[...] \\ j_z(n,n') &= \alpha\epsilon(-i\sigma P)(33+44)\exp[...] \end{aligned}\right\}, \quad (17)$$

where
$$33+44 = r^{-i\delta-1}\exp[-(b+ik)r] \times$$
$$\times\{B(j+l)P_{j-1}^{l}\exp(-il\phi) + AP_{j-1}^{l+1}\exp[-i(l+1)\phi]\}. \quad (18)$$

The current is, to this order, parallel to the electric force in the wave. The value of ρ, not written down, is useful in verifying the equation of continuity, and hence in checking the work.

We are considering in effect the asymptotic value of $33+44$, and hence can replace $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, when operating on j, by

$$-ik(1-\mu^2)^{\frac{1}{2}}\cos\phi+2\pi i\nu\sigma/c,\quad -ik(1-\mu^2)^{\frac{1}{2}}\sin\phi,\quad -ik\mu+2\pi i\nu\kappa/c \tag{19}$$

respectively. Since the product of any tesseral harmonic with μ or $(1-\mu^2)^{\frac{1}{2}}\exp(\pm i\phi)$ is a sum of tesseral harmonics, each component of curl j is a sum of terms of the form

$$J=r^{-i\delta-1}\exp[-(b+ik)r]\times Y_n(\theta,\phi)\exp[-2\pi i\nu' t+2\pi i\nu(\sigma x+\kappa z)/c]. \tag{20}$$

To find the radiation at a distant point $(\sigma' r',\tau' r',\kappa' r')$, where $\sigma'^2+\tau'^2+\kappa'^2=1$, multiply as usual by $\exp[-2\pi i\nu'(\sigma' x+\tau' y+\kappa' z)/c]$, and H is the static potential of the curl j so modified. Thus if we write

$$\left.\begin{aligned} 2\pi(\nu\sigma-\nu'\sigma')c^{-1}=\rho'\sin\theta'\cos\phi',\qquad 2\pi(-\nu'\tau')c^{-1}=\rho'\sin\theta'\sin\phi',\\ 2\pi(\nu\kappa-\nu'\kappa')c^{-1}=\rho'\cos\theta',\end{aligned}\right\} \tag{21}$$

the part of the magnetic force arising from (20) is

$$\frac{\exp(2\pi i\nu' r'/c)}{r'}\int J' r^2\,dr\sin\theta\,d\theta d\phi,$$

where $\quad J'=r^{-i\delta-1}\exp[-2\pi i\nu' t-(b+ik)r]\times$

$$\times Y_n(\theta,\phi)\exp i\rho' r[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')].$$

The result of integrating the last line with respect to the solid angle is† $(2\pi)^{\frac{3}{2}}i^n(\rho' r)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho' r)Y_n(\theta',\phi')$; and using Hankel's integral‡

$$\int\limits_0^\infty r^m\exp(-ar)J_n(br)\,dr$$

$$=\frac{(\frac{1}{2}b)^n\Gamma(m+n+1)}{a^{m+n+1}\Gamma(n+1)}\,{}_2F_1\!\left(\frac{m+n+1}{2},\frac{m+n+2}{2};n+1;-\frac{b^2}{a^2}\right),$$

the conditions of validity of which are all satisfied, we find H in terms of a hypergeometric function of argument

$$x=[\rho'/(k-ib)]^2, \tag{22}$$

which is small when x is large, and large in the neighbourhood of $x=1$. Hence there is a sharp maximum of H in the neighbourhood of

$$\rho'=k. \tag{23}$$

Since the function ${}_2F_1(a,b;c;x)$, in which $R(a+b-c)>0$, is approximately $\Gamma(c)\Gamma(a+b-c)/\Gamma(a)\Gamma(b)(1-x)^{a+b-c}$ when x is near 1, we find that when (23) is nearly satisfied

$$H=\frac{(2\pi)2^{1-i\delta}\Gamma(1-i\delta)}{(ik)^{2-i\delta}(1-x)^{1-i\delta}}\frac{\exp 2\pi i\nu'(r'/c-t)}{r'}Y_n(\theta',\phi'). \tag{24}$$

Let R be a direction (σ',τ',κ') in which radiation is observed, and E a direction (θ',ϕ') connected with it by equations (21). Then since

† L. Gegenbauer, *Wien. Ber.* 85 (2) (1882), 498; E. T. Whittaker, *Math. Ann.* 57 (1903), 348. ‡ H. Hankel, *Math. Ann.* 8 (1875), 467.

n has disappeared from (24), except as a suffix of Y_n, any addition of terms which converts $Y_n(\theta,\phi)$ in (20) into $f(\theta,\phi)$ converts $Y_n(\theta',\phi')$ in (24) into $f(\theta',\phi')$. Composite wave functions are formed by multiplying $\psi(n')^*$ by coefficients $a_{n'}$ and adding, where n' is a symbol for the pair of quantum numbers j, l. If the coefficients are chosen to make $\sum_{n'} a_{n'} j(n,n')$ appreciable only at points on some line E, the radiation is appreciable only in the corresponding direction R, with the usual limitation of precision in quantum mechanics. Since $j(n,n')$ contains, in the quantity $33+44 \equiv \psi_3(n)\psi_3(n')^*+\psi_4(n)\psi_4(n')^*$, two distinct constants A, B, it would be plausible to assume that $\sum a_{n'}\psi_3(n')^*$ and $\sum a_{n'}\psi_4(n')^*$ are each appreciable only on the line E. A better argument is that the components of j only contain a single function $33+44$ by an accident of approximation, which is not repeated when α is large. Thus to make ρ and j zero in (6) in general requires all four components of one wave function to vanish.

To find the volume and current density at great distances of a number of superposed recoil functions, all of energy βmc^2, change the notation and write $\psi(n)$ for one of them, using the reduction formulae of tesseral harmonics to eliminate $j+l$ and $j-l$. Thus

$$\psi_1(n) = \frac{iF_j(\mu P_j^{l+1} - P_{j-1}^{l+1})\exp i(l+1)\phi}{(1-\mu^2)^{\frac{1}{2}}\exp i\phi}, \qquad \psi_2(n) = iF_j P_j^{l+1}\exp i(l+1)\phi,$$

$$\psi_3(n) = \frac{G_j(\mu P_{j-1}^{l+1} - P_j^{l+1})\exp i(l+1)\phi}{(1-\mu^2)^{\frac{1}{2}}\exp i\phi}, \qquad \psi_4(n) = G_j P_{j-1}^{l+1}\exp i(l+1)\phi.$$

Using (13), we find that the mutual volume and current density of any two states of equal energy is given by

$$\frac{j_{x+iy}(n,n')}{(1-\mu^2)^{\frac{1}{2}}\exp i\phi} = \frac{j_{x-iy}(n,n')}{(1-\mu^2)^{\frac{1}{2}}\exp(-i\phi)} = \frac{j_z(n,n')}{\mu} = \frac{(\beta^2-1)^{\frac{1}{2}}\rho(n,n')}{\beta}.$$

Hence the total current density of any composite state of energy βmc^2 is ultimately radial and of magnitude $\rho v/c$, where $\beta = (1-v^2/c^2)^{-\frac{1}{2}}$.† Neglecting products of the small wave functions ψ_1, ψ_2, $\psi_3 = \psi_4 = 0$ makes the current vanish. Hence the direction E referred to above is that of the recoil electron, and (21) and (23) are the usual equations expressing conservation of momentum in the corpuscular theory. The polarization is evidently 'classical', that is that due to a Hertzian oscillator whose axis points along the electric force in the wave, as found by Waller, and by Klein and Nishina at low frequencies.

† A different result has been obtained by F. Sauter, *Ann. d. Phys.* 9 (1931), 246.

A general method of calculating polarizations, apart from any special form of (17), is to take the E direction as axis of z and the R direction as $(\sigma', 0, \kappa')$, where $\nu\sigma = \nu'\sigma'$. We need then only retain terms $l = -1$ and $l = 0$, putting $P_n^m = 1$ when $m = 0$ and zero otherwise. Application to (17), with superposition of wave functions, introduces two constants, $A \sum a_{n'}$ corresponding to $l = -1$ and $B \sum j a_{n'}$ corresponding to $l = 0$. We find that

$$H_x : H_y : H_z = \kappa' Q : -(\kappa\kappa' + \sigma\sigma')P : -\sigma' Q,$$

the classical rule.

It remains to justify the neglect of the term eV. Taking the value of $\delta\psi$ in (15) as a first approximation, the next is found by writing $f \mp (eV/h\nu)\delta\psi^{\pm} = f \mp (c\gamma/2\pi\nu)\delta\psi^{\pm}/r$ instead of f in (4). Thus $\delta\psi_1$ is changed in the ratio of $1-\gamma^2/2br$ to 1, which, since the original radial integrals converge near $r = 0$ like $\int r\,dr$, has but a small effect on them.

We now try to solve equations (4) when α is large, treating bound and recoil electrons differently. For the bound electron, the terms independent of ν give four equations of the type $\delta\psi_1 + \kappa\delta\psi_3 + \sigma\delta\psi_4 = f_1$; but since the determinant is of rank 2, $\delta\psi$ cannot be of order f. Begin by neglecting f and solving the homogeneous equations in the form

$$\delta\psi_1^+ = C_1, \quad \delta\psi_2^+ = C_2, \quad \delta\psi_3^+ = -\kappa C_1 - \sigma C_2, \quad \delta\psi_4^+ = -\sigma C_1 + \kappa C_2,$$

where C_1, C_2 are so far two arbitrary functions of x, y, z. Write to a next approximation

$$\delta\psi_1^+ = C_1 + \phi_1/\nu, \qquad \delta\psi_2^+ = C_2 + \phi_2/\nu,$$
$$\delta\psi_3^+ = -\kappa C_1 - \sigma C_2 + \phi_3/\nu, \qquad \delta\psi_4^+ = -\sigma C_1 + \kappa C_2 + \phi_4/\nu,$$

and equate to zero the terms of order $1/\nu$, regarding f as of order $1/\nu$. This gives four equations of the type

$$\phi_1 + \kappa\phi_3 + \sigma\phi_4 = f_1 - \left[\frac{h\nu(n) + eV + mc^2}{h\nu}\right] C_1 -$$
$$- \frac{c}{2\pi i\nu}\frac{\partial}{\partial z}(-\kappa C_1 - \sigma C_2) - \frac{c}{2\pi i\nu}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(-\sigma C_1 + \kappa C_2).$$

The conditions of solubility are that the tetrad of functions on the right shall be orthogonal to the two independent solutions $(1, 0, -\kappa, -\sigma)$, $(0, 1, -\sigma, \kappa)$ of the transposed homogeneous equations. The two conditions separate C_1 and C_2 and give

$$2\left[\frac{h\nu(n)+eV}{h\nu}-\frac{c}{2\pi i\nu}\left(\sigma\frac{\partial}{\partial x}+\kappa\frac{\partial}{\partial z}\right)\right]C_1=f_1-\kappa f_3-\sigma f_4,$$

$$2\left[\frac{h\nu(n)+eV}{h\nu}-\frac{c}{2\pi i\nu}\left(\sigma\frac{\partial}{\partial x}+\kappa\frac{\partial}{\partial z}\right)\right]C_2=f_2-\sigma f_3+\kappa f_4,$$

each of which can be solved by quadrature. It is sufficient to put $h\nu(n) = mc^2$ and neglect V and the differential terms, so that

$$\left.\begin{array}{ll}\delta\psi_1^+ = \frac{1}{2}\alpha(f_1-\kappa f_3-\sigma f_4), & \delta\psi_2^+ = \frac{1}{2}\alpha(f_2-\sigma f_3+\kappa f_4)\\ \delta\psi_3^+ = \frac{1}{2}\alpha(-\kappa f_1-\sigma f_2+f_3), & \delta\psi_4^+ = \frac{1}{2}\alpha(-\sigma f_1+\kappa f_2+f_4)\end{array}\right\}. \quad (25)$$

The recoil electron differs in that the values of β which we have to consider are close to α, the corpuscular theory giving approximately $\beta = \alpha-\cos\theta/(1-\cos\theta)$ for an angle of scattering θ, and

$$\nu' = \nu-ck/2\pi. \quad (26)$$

Hence the diagonal terms in equation (4) are of order $1/\nu$, and the asymptotic values of $\delta\psi$ can be calculated and are found to be of order f as against αf in (25). Something of the kind might be expected from the relatively smaller perturbation of a fast electron in classical mechanics. Hence we write instead of (7)

$$\psi_r\psi_s'^* = \delta\psi_r^+\psi_s(n')^*\exp\left(-2\pi i\nu' t+2\pi i\nu\frac{\sigma x+\kappa z}{c}\right), \quad (27)$$

with $\delta\psi_r^+$ taken from (25). The theory proceeds as before. We have the conservation equations (21) but no simple rule of polarization, the scattered radiation consisting of two elliptic waves for which the theory of Klein and Nishina prepares us.

The calculations of this paper are opposed to the view that wave mechanics is incompetent to express the idea that the directions of scattering of electron and light quantum are correlated, which seems to have been favoured by Darwin.† Darwin's note contains an interesting suggestion allied closely with the theory that a photon is a particle of negligible charge and mass, which may have some bearing on the fact that a corpuscular Compton effect can be extracted from an unquantized ether. Though such a particle may acquire any energy $h\nu$ and the corresponding momentum $h\nu/c$, it cannot be quantized by a nucleus since the quantized frequencies tend to zero with e and m.

† C. G. Darwin, *Nature*, 123 (1929), 203.

THE LOWER BOUNDS OF INDEFINITE HERMITIAN QUADRATIC FORMS

By A. OPPENHEIM (*Singapore*)

[Received 20 May 1931]

1. THE variables which occur in this note are integers, real or complex; the type in each case will be clear from the context. The coefficients will be real or complex numbers.

Suppose that $h = h_n$ is an Hermitian quadratic form in the complex integral variables $x_1,..., x_n$ of assigned non-zero determinant Δ. Our problem is to find the upper bound of $L(h)$, where $L(h)$ is the lower bound of the absolute values of h for all sets of complex integers other than $(0, 0,..., 0)$. We confine ourselves to indefinite forms.

Meyer* has shown that any real indefinite quinary form represents zero if its coefficients be rational. Since any n-ary indefinite Hermitian form, h_n, may be regarded as a special type of a $2n$-ary real indefinite quadratic form, it follows that h_n represents zero properly and so $L(h_n) = 0$ if the coefficients of h_n are rational complex and $n \geqslant 3$. We confine ourselves therefore to binary forms.

Since we require an upper bound for $L(h)$ and $L \geqslant 0$, we may suppose that $L > 0$. We also assume that h actually represents L, which is certainly the case if the coefficients of h are rational complex. A slight extension of the arguments employed show that the results hold even when L is not attained.

Since L is necessarily represented properly, a complex integral transformation of determinant ± 1 or $\pm i$ enables us to take

$$\frac{h}{L} = x\bar{x} + \beta x\bar{y} + \bar{\beta}\bar{x}y + \gamma y\bar{y},$$

where
$$\beta\bar{\beta} - \gamma = \frac{\Delta}{L^2} = D > 0$$

and the bar as usual denotes the conjugate complex number.

If the real and imaginary parts of z be z_1 and z_2, we obtain the real form

$$\phi = (x_1 + \beta_1 y_1 + \beta_2 y_2)^2 + (x_2 - \beta_2 y_1 + \beta_1 y_2)^2 - D(y_1^2 + y_2^2)$$

in the real integral variables x_1, x_2, y_1, y_2 of absolute minimum 1 and

* See L. J. Mordell, *Jour. für Math.*, 164 (1931), 40–9, for a simple proof and references to the literature.

determinant D^2. By a known theorem* we must have either $D^2 = \frac{9}{4}$ or else $D^2 \geqslant \frac{17}{4}$. We can, however, obtain much more information about this special type of quaternary form ϕ by simple and elementary arguments based on the hypothesis that $|\phi| \geqslant 1$ for integers $x_1, \ldots, y_2$ not all zero.

It will conduce to economy to give first some known results about definite and indefinite real binary forms. We apply these results to indefinite ternary forms of a certain kind in § 3. Our main theorem (§ 4) is obtained by application of the theorem of § 3.

2. LEMMA 1. *If the definite binary form* $(x+\alpha y)^2+\Delta y^2$ $(\Delta > 0)$, *has minimum unity, then* $4\Delta \geqslant 3$.

This is classical. Take $y = 1$ and choose x so that $|x+\alpha y| \leqslant \frac{1}{2}$. Then $\Delta+\frac{1}{4} \geqslant 1$.

LEMMA 2. *Suppose that the indefinite binary form* $(x+\alpha y)^2-\Delta y^2$, *where* $\Delta > 0$, *has absolute minimum unity. Then necessarily*

$$\Delta = \tfrac{5}{4},\ \alpha \equiv \tfrac{1}{2}; \qquad \Delta = 2,\ \alpha \equiv 0; \qquad \Delta = 2{\cdot}21,\ \alpha \equiv \tfrac{1}{10};$$

or else $\Delta > 2{\cdot}21$. *The congruences are modulo* 1.

The results of this lemma are well known. They form part of Markoff's theorem on the minima of indefinite binary quadratic forms. This theorem was proved in different fashion by Frobenius and by Remak.† The short proof which follows of the results of Lemma 2 may be of interest.

By hypothesis, for integers x and y not both zero, one of the inequalities

$$\Delta y^2 \leqslant (x+\alpha y)^2-1, \qquad \Delta y^2 \geqslant (x+\alpha y)^2+1$$

must be true. Denote the first by $(P; x, y)$ and the second by $(N; x, y)$. We may plainly suppose that $0 \leqslant \alpha \leqslant \frac{1}{2}$.

Now $(P; 0, 1)$ contradicts $\Delta > 0$. Therefore $(N; 0, 1)$ is true.

If $(P; 1, 1)$, then we have in succession

$$(1+\alpha)^2-1 \geqslant \Delta \geqslant \alpha^2+1; \qquad \alpha \geqslant \tfrac{1}{2}; \qquad \alpha = \tfrac{1}{2}; \qquad \Delta = \tfrac{5}{4},$$

the first result of Lemma 2. Discarding this case we must have $(N; 1, 1)$ which is stronger than $(N; 0, 1)$.

If $(P; 3, -2)$, then again in succession

$$(3-2\alpha)^2-1 \geqslant 4\Delta \geqslant 4+4(1+\alpha)^2; \qquad 0 \geqslant 20\alpha; \qquad \alpha = 0,\ \Delta = 2,$$

* A. Oppenheim, *Annals of Math.*, 32 (1931), 271–98 (272).

† R. Remak, *Math. Ann.*, 92 (1924), 155–82, and references there given. For an exposition of Markoff's proof, see L. E. Dickson, *Studies in the Theory of Numbers*, Chicago (1930).

the second result of Lemma 2. Discarding this case, we must have $(N; +3, -2)$ and also $(N; 1, 1)$, so that

$$4\Delta \geqslant \max[1+(3-2\alpha)^2, 4+4(1+\alpha)^2].$$

Now the two quadratics on the right are equal to 8·84 when $10\alpha = 1$. Also the first quadratic decreases and the second increases when α increases from 0 to $\frac{1}{2}$. Hence, $4\Delta \geqslant 8{\cdot}84$ and equality can occur, if and only if $10\alpha = 1$.

The rest of Lemma 2 follows immediately.

3. In the form ϕ of § 1 put y_2 or y_1 equal to zero. We obtain an indefinite ternary form f of the type

$$f = (x+\alpha z)^2+(y+\beta z)^2-\Delta z^2 \qquad (\Delta > 0),$$

of absolute minimum unity. We apply Lemmas 1 and 2 to find the possible values of Δ. We may suppose that

$$0 \leqslant \alpha \leqslant \tfrac{1}{2}, \qquad 0 \leqslant \beta \leqslant \tfrac{1}{2}.$$

Put $y = ut$, $z = vt$ where u and v are integers not both zero, and t is an integer. We obtain a binary quadratic form in x and t,

$$(x+\alpha vt)^2+\{(u+\beta v)^2-\Delta v^2\}t^2,$$

a section of f, which has absolute minimum unity. By Lemmas 1 and 2, consequently, one of the following mutually exclusive results must hold:

$$\begin{array}{lll}
(B,u,v;1) & \frac{3}{4}+\Delta v^2 \leqslant (u+\beta v)^2; & \\
(B,u,v;2) & \Delta v^2 = \frac{5}{4}+(u+\beta v)^2, & \alpha v \equiv \frac{1}{2}; \\
(B,u,v;3) & \Delta v^2 = 2+(u+\beta v)^2, & \alpha v \equiv 0; \\
(B,u,v;4) & \Delta v^2 = 2{\cdot}21+(u+\beta v)^2, & \alpha v \equiv \frac{1}{10}; \\
(B,u,v;5) & \Delta v^2 > 2{\cdot}21+(u+\beta v)^2. &
\end{array}$$

A similar set $(A, u, v; i)$ holds with α and β interchanged.

$(B, 0, 1; 1)$ contradicts $\Delta > 0$. If $(B, 0, 1; 2)$, then $\alpha = \frac{1}{2}$ and $\Delta \leqslant \frac{5}{4}+\frac{1}{4}$, so that $(A, 0, 1; 2)$ must hold. We obtain the form

$$(x+\tfrac{1}{2}z)^2+(y+\tfrac{1}{2}z)^2-\tfrac{3}{2}z^2, \qquad \Delta = \tfrac{3}{2},\ \alpha = \beta = \tfrac{1}{2}.$$

Apart from this case, we must have $\Delta \geqslant 2+\beta^2$, $\Delta \geqslant 2+\alpha^2$ from $(B, A, 0, 1; i \geqslant 3)$. Since $(B, A, 1, 1; 1)$ yield $\Delta \leqslant \frac{3}{2}$, we must have $(B, A, 1, 1; i \geqslant 2)$.

$(B, 1, 1; 2)$ gives $\alpha = \frac{1}{2}$ and $\Delta \leqslant \frac{7}{2}$; then $\alpha = \frac{1}{2}$ and $(A, 1, 1; 2)$ give $\Delta = \frac{7}{2}$ and $\beta = \frac{1}{2}$. We obtain the form

$$(x+\tfrac{1}{2}z)^2+(y+\tfrac{1}{2}z)^2-\tfrac{7}{2}z^2; \qquad \Delta = \tfrac{7}{2},\ \alpha = \beta = \tfrac{1}{2}.$$

On the other hand $\alpha = \frac{1}{2}$ and $(A, 1, 1; 3)$ yield $4\Delta = 17$, $\beta = 0$, and so $4\Delta = 9$ by $(B, 1, 1; 2)$, a contradiction. Hence $(A, 1, 1; i \geqslant 4)$ must hold, and therefore

$$\Delta \geqslant 2{\cdot}21 + 1{\cdot}5^2 = 4{\cdot}46.$$

We must now consider $(B, 1, 1; 3)$ whence $\alpha = 0$. If $(A, 1, 1; 3)$, then $\beta = 0$, $\Delta = 3$, and we obtain the form

$$x^2 + y^2 - 3z^2, \qquad \Delta = 3,\ \alpha = \beta = 0.$$

In any other case we have $(A, B, 1, 1; i \geqslant 4)$, and therefore

$$\Delta \geqslant \max\{2{\cdot}21 + (1+\alpha)^2,\ 2{\cdot}21 + (1+\beta)^2\} \geqslant 3{\cdot}21.$$

Suppose therefore that $\Delta \geqslant 3{\cdot}21$. Since $(A, B, -2, 1; 1)$ give $\Delta \leqslant 3 + \frac{1}{4}$, they are compatible only with $(A, B, 1, 1; i \geqslant 4)$ by the preceding analysis. Hence

$$2 + (1+\beta)^2 < \Delta \leqslant (2-\beta)^2 - \tfrac{3}{4}$$

by $(B, -2, 1; 1)$, so that $24\beta < 1$, and similarly, since $(B, -2, 1; 1)$ involves also $(A, -2, 1; 1)$, $24\alpha < 1$. But now

$$0 < f(0, -2, 1) = \alpha^2 + (2-\beta)^2 - \Delta < 1,$$

contradicting $|f| \geqslant 1$. Thus $(A, B; -2, 1; 1)$ are excluded if $\Delta \geqslant 3{\cdot}21$.

If $(B, -2, 1; 2)$, then $\alpha = \frac{1}{2}$. If also $(A, -2, 1; 2)$, then $\Delta = \frac{7}{2}$, $\beta = \frac{1}{2}$. We obtain the form given by $(A, B; 1, 1; 2)$. If, however, $(A, -2, 1; 3)$ then $4\Delta = 17$, $\beta = 0$, and so $4\Delta = 13$ by $(B, -2, 1; 2)$, a contradiction. Hence $(A, -2, 1; i \geqslant 4)$ hold, and so

$$\Delta \geqslant 2{\cdot}21 + 2{\cdot}25 = 4{\cdot}46.$$

If $(A, -2, 1; 2)$ holds, the roles of α and β are interchanged.

There remain finally $(A, B, -2, 1; i \geqslant 3)$ which give $\Delta \geqslant 4{\cdot}46$. Actually the sign $\geqslant$ may be replaced by $>$.

It is essential to observe that the three forms obtained each have minimum unity. Summing up, we obtain the following:

THEOREM. *If the indefinite ternary form*

$$f = (x+\alpha z)^2 + (y+\beta z)^2 - \Delta z^2 \qquad (\Delta > 0),$$

has absolute minimum unity, then necessarily

$$\Delta = \tfrac{3}{2}, \alpha \equiv \beta \equiv \tfrac{1}{2}; \qquad \Delta = 3, \alpha \equiv \beta \equiv 0; \qquad \Delta = \tfrac{7}{2}, \alpha \equiv \beta \equiv \tfrac{1}{2};$$

or else $\Delta \geqslant 4{\cdot}46$. *The congruences are modulo* 1.

We may enunciate this theorem differently: *if an indefinite ternary form f has (negative) determinant $-\Delta$, absolute lower bound $m > 0$, and*

represents the binary form $m(x^2+y^2)$, *then*

either $m^3 = \frac{2}{3}\Delta$ *and* $f \sim m[(x+\frac{1}{2}z)^2+(y+\frac{1}{2}z)^2-\frac{3}{2}z^2]$,

or $m^3 = \frac{1}{3}\Delta$ *and* $f \sim m[x^2+y^2-3z^2]$,

or $m^3 = \frac{2}{7}\Delta$ *and* $f \sim m[(x+\frac{1}{2}z)^2+(y+\frac{1}{2}z)^2-\frac{7}{2}z^2]$,

or else $m^3 \leqslant \frac{50}{223}\Delta$.

4. We apply now the theorem of § 3 to the quaternary form ϕ of § 1. Put $y_1 = 0$ or $y_2 = 0$. We obtain two equivalent indefinite ternary forms of absolute minimum unity and determinant $-D$. Application of § 3 yields the quaternary forms:

$$(x_1+\tfrac{1}{2}y_1+\tfrac{1}{2}y_2)^2+(x_2-\tfrac{1}{2}y_1+\tfrac{1}{2}y_2)^2-\tfrac{3}{2}(y_1^2+y_2^2), \qquad D = \tfrac{3}{2};$$

$$x_1^2 + x_2^2 - 3(y_1^2+y_2^2), \qquad D = 3;$$

$$(x_1+\tfrac{1}{2}y_1+\tfrac{1}{2}y_2)^2+(x_2-\tfrac{1}{2}y_1+\tfrac{1}{2}y_2)^2-\tfrac{7}{2}(y_1^2+y_2^2), \qquad D = \tfrac{7}{2};$$

each of absolute minimum unity; or else $D \geqslant 4{\cdot}46$.

Returning to our indefinite binary Hermitian form h, we have finally the theorem

THEOREM. *If L is the lower bound of an indefinite binary Hermitian form h of (negative) determinant $-\Delta$, then*

either $L^2 = \frac{2}{3}\Delta$ *and* $h \sim L\left(x\bar{x}+\frac{1+i}{2}x\bar{y}+\frac{1-i}{2}\bar{x}y-y\bar{y}\right)$;

or $L^2 = \frac{1}{3}\Delta$ *and* $h \sim L(x\bar{x}-3y\bar{y})$;

or $L^2 = \frac{2}{7}\Delta$ *and* $h \sim L\left(x\bar{x}+\frac{1+i}{2}x\bar{y}+\frac{1-i}{2}\bar{x}y-3y\bar{y}\right)$;

or else $L^2 \leqslant \frac{50}{223}\Delta$.

GENERALIZED SOLUTIONS OF SOME OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

By H. S. RUSE (*Edinburgh*)

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THE present paper contains an expression in tensor form of some theorems due to A. R. Forsyth* concerning solutions of the partial differential equation

$$\nabla^2 V + \kappa^2 V = 0, \tag{1}$$

where ∇^2 is the Laplacian operator $\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}+\ldots+\dfrac{\partial^2}{\partial x_n^2}$ in n variables, and κ is a constant. The theorems are given in § 2. Equation (1) includes a number of equations important in mathematical physics; for example, if $n = 4$, $\kappa = 0$ and $ct\sqrt{-1}$ is written for x_4, the classical wave-equation is obtained.

§ 1 contains some general remarks about the ideas underlying the subsequent work; §§ 3 and 4 contain the generalizations of Forsyth's theorems, §§ 5 and 6 being devoted to proofs.

1. Tensor differential equations

The absolute differential calculus of Ricci and Levi-Civita has proved a powerful instrument in the development of a number of branches of pure mathematics, a fact which suggests the desirability of developing an 'absolute integral calculus'. This idea leads one to inquire whether it is possible to define a unique operation inverse to that of covariant differentiation. The answer is fairly obviously in the negative, since an attempt to define such an operation leads almost immediately to the necessity of solving partial differential equations.

The most fruitful way of beginning the development of an absolute integral calculus would therefore seem to be to study tensor differential equations and their solutions. Perhaps the simplest of such differential equations are those involving the tensor generalization of the Laplacian operator ∇^2, the easiest method of finding solutions

* *Messenger of Math.*, 27 (1897), 99.

being to express in tensor form solutions already known for the non-tensorized equation. This is in effect the method adopted below.

The method has, however, certain serious limitations. The expression of formulae in tensor form implies the introduction of an associated riemannian geometry, and therefore the selection of an appropriate metric. As a rule it is necessary in the first place to choose the metric of a flat space, further investigation being required to show whether the solutions obtained are valid in a non-flat space. This investigation is often a matter of no little difficulty. Indeed, in the case of the theorems of the present paper, it has been possible to obtain results only for flat spaces and for spaces of constant curvature.

The difficulty arises in this way. In the differential equation $\nabla^2 V + \kappa^2 V = 0$ the variables are assumed to be rectangular cartesian, the associated metric being therefore defined by

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2.$$

Now rectangular cartesian coordinates are a special case of the so-called normal coordinates of general riemannian geometry. But if $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ defines the metric of a general riemannian space, then the Riemann-Veblen normal coordinates (y^μ) (which in a flat space are cartesian, though in general non-rectangular) are defined as functions of the coordinates (x^μ) by the equation*

$$y^\mu = -\bar{g}^{\mu\nu} \frac{\partial \Omega}{\partial \bar{x}^\nu},$$

where Ω is one-half of the square of the geodesic distance between a fixed point $(\bar{x}^\mu)$ and the variable point (x^μ), and $\bar{g}^{\mu\nu}$ is the value of $g^{\mu\nu}$ at $(\bar{x}^\mu)$. Consequently any generalization to tensors of a solution of the given partial differential equation will almost inevitably involve the derivatives of the function Ω thus defined. Now in the case of a flat space this function possesses certain very simple properties, similar ones being enjoyed† in a space of constant curvature K by the function $Q = \cos(2K\Omega)^{\frac{1}{2}}$. But no such simple properties appear to exist in the case of a general riemannian space, and hence arises the difficulty of obtaining completely generalized solutions of the differential equations.

* Ruse, *Proc. London Math. Soc.*, 32 (1931), 90, eq. (13).

† See *Proc. London Math. Soc.*, 31 (1930), 225, and *Quart. J. of Math.* (Oxford), 1 (1930), 146.

2. Forsyth's theorems

The principal results of the paper quoted above may be summarized as follows:

Let $p_1, p_2,..., p_n$ denote n arbitrary functions of a variable u, subject to the single condition

$$p_1^2+p_2^2+...+p_n^2 = 0, \tag{2.1}$$

and let u be determined as a function of n independent variables $x_1, x_2,..., x_n$ by the equation

$$au = x_1 p_1+x_2 p_2+...+x_n p_n, \tag{2.2}$$

where a is any constant. Let also

$$\eta = x_1 p_1'+x_2 p_2'+...+x_n p_n', \tag{2.3}$$

$$\theta = p_1'^2+p_2'^2+...+p_n'^2, \tag{2.4}$$

the accents denoting differentiation with respect to u. Then

I. If $f(u)$ is an arbitrary function of u,

$$V = f(u) \tag{2.5}$$

is a solution of the partial differential equation

$$\nabla^2 V = 0, \tag{2.6}$$

where, as above, ∇^2 is the Laplacian operator $\sum\limits_1^n \frac{\partial^2}{\partial x_r^2}$.

II. If $\phi(u)$ and $\psi(u)$ are arbitrary functions of u, then

$$V = \frac{\phi(u)}{a-\eta}\exp\left(\frac{i\kappa\eta}{\theta}\right)+\frac{\psi(u)}{a-\eta}\exp\left(-\frac{i\kappa\eta}{\theta}\right) \tag{2.7}$$

is a solution of the equation

$$\nabla^2 V+\kappa^2 V = 0. \tag{2.8}$$

When $\kappa = 0$ it follows that, if $\chi(u)$ is an arbitrary function of u, then $V = \chi(u)(a-\eta)^{-1}$ is a solution of $\nabla^2 V = 0$, and since this equation is linear it follows by I that $V = f(u)+\chi(u)(a-\eta)^{-1}$ is also a solution. A remark similar to this applies to the formulae obtained below, which are a generalization of I and II.

The truth of the above theorems, the proofs of which are remarkably simple in view of their great generality, will be assumed in the following paragraphs.

3. Tensor formulae for a flat space

Let
$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{3.1}$$
define the metric of an n-dimensional flat (euclidean) space.* Then the partial differential equation of which solutions are sought is
$$g^{\mu\nu}V_{\mu\nu}+\kappa^2 V = 0, \tag{3.2}$$
where
$$V_{\mu\nu} = \frac{\partial^2 V}{\partial x^\mu \partial x^\nu} - \{\mu\nu, \alpha\}\frac{\partial V}{\partial x^\alpha}.$$

Equation (3.2) is a scalar equation, since $V_{\mu\nu}$ is the second covariant derivative of the scalar V.

Let $(\bar{x}^\mu) \equiv (\bar{x}^1, \bar{x}^2,..., \bar{x}^n)$ be any given fixed point of the space and let Ω, as above, represent one-half of the square of the geodesic distance between $(\bar{x}^\mu)$ and a variable point (x^μ), so that Ω is a function of the x's and of the $\bar{x}$'s. Let $\bar{g}^{\mu\nu}$ be the value at $(\bar{x}^\mu)$ of the contravariant tensor $g^{\mu\nu}$, and write $\Omega^{(\mu)}$ for $\bar{g}^{\mu\alpha}\partial\Omega/\partial\bar{x}^\alpha$.

Now suppose that $p_{(\alpha)} \equiv [p_{(1)}, p_{(2)},..., p_{(n)}]$ denote† n arbitrary functions of a variable u, subject to the single condition
$$\bar{g}^{\mu\nu}p_{(\mu)}p_{(\nu)} = 0, \tag{3.3}$$
(there being a summation with respect to μ and ν), and let u be determined as a scalar function of the n independent variables (x^μ) by the equation
$$au = -\Omega^{(\alpha)}p_{(\alpha)} \tag{3.4}$$
where a is any constant. Let also
$$\eta = -\Omega^{(\alpha)}p'_{(\alpha)} \tag{3.5}$$
and
$$\theta^2 = \bar{g}^{\mu\nu}p'_{(\mu)}p'_{(\nu)}, \tag{3.6}$$
the accents denoting, as always hereafter, differentiation with respect to u.

Then the following theorems are true:

THEOREM A. *If $f(u)$ is an arbitrary function of u, then*
$$V = f(u) \tag{3.7}$$
is a solution of the partial differential equation
$$g^{\mu\nu}V_{\mu\nu} = 0. \tag{3.8}$$

* No assumption is being made as to the definiteness or indefiniteness of the quadratic form in (3.1).

† The suffixes are written in brackets to indicate that they are not ordinary tensor-suffixes.

THEOREM B. *If $\phi(u)$ and $\psi(u)$ are arbitrary functions of u, then*

$$V = \frac{\phi(u)}{a-\eta}\exp\left(\frac{i\kappa\eta}{\theta}\right)+\frac{\psi(u)}{a-\eta}\exp\left(-\frac{i\kappa\eta}{\theta}\right) \tag{3.9}$$

is a solution of the partial differential equation

$$g^{\mu\nu}V_{\mu\nu}+\kappa^2 V = 0. \tag{3.10}$$

The proof is simple if it is observed that all the equations involved, namely (3.3) to (3.10), are *scalar* relations. Hence, if the theorems are true for any one system of coordinates, they will be true for all sets derivable from them by point-transformations. Hence, since the space is flat, it is necessary to demonstrate the truth of the theorems only for the case when the fundamental quadratic form is

$$ds^2 = (dx^1)^2+(dx^2)^2+...+(dx^n)^2. \tag{3.11}$$

For this

$$\left.\begin{aligned} g_{\mu\nu} = g^{\mu\nu} = \bar{g}_{\mu\nu} = \bar{g}^{\mu\nu} &= 1 \text{ if } \mu = \nu \\ &= 0 \text{ if } \mu \neq \nu \end{aligned}\right\}, \tag{3.12}$$

and

$$\Omega = \tfrac{1}{2}[(x^1-\bar{x}^1)^2+(x^2-\bar{x}^2)^2+...+(x^n-\bar{x}^n)^2], \tag{3.13}$$

whence

$$\begin{aligned} \Omega^{(\mu)} &= \bar{g}^{\mu\alpha}\partial\Omega/\partial\bar{x}^\alpha \qquad (\mu = 1, 2,..., n) \\ &= \partial\Omega/\partial\bar{x}^\mu \qquad \text{by (3.12)} \\ &= -(x^\mu-\bar{x}^\mu) \qquad \text{by (3.13)}. \end{aligned}$$

So (3.3), (3.4), (3.5), and (3.6) become respectively

$$p^2_{(1)}+p^2_{(2)}+...+p^2_{(n)} = 0 \tag{3.14}$$

$$au = (x^1-\bar{x}^1)p_{(1)}+(x^2-\bar{x}^2)p_{(2)}+...+(x^n-\bar{x}^n)p_{(n)} \tag{3.15}$$

$$\eta = (x^1-\bar{x}^1)p'_{(1)}+(x^2-\bar{x}^2)p'_{(2)}+...+(x^n-\bar{x}^n)p'_{(n)} \tag{3.16}$$

$$\theta^2 = p'^2_{(1)}+p'^2_{(2)}+...+p'^2_{(n)}, \tag{3.17}$$

and the partial differential equations (3.8) and (3.10) reduce to $\nabla^2 V = 0$ and $\nabla^2 V+\kappa^2 V = 0$ respectively.

By I and II of § 2 the truth of theorems A and B is now obvious. The fact that the differences $x^1-\bar{x}^1$, $x^2-\bar{x}^2$, etc., appear in (3.15) and (3.16), instead of simply the x^1, x^2,... in (2.2) and (2.3), is of no consequence since the $\bar{x}$'s are constants and the replacement of the x's by the differences $x-\bar{x}$ amounts therefore to a mere change of origin.

4. Corresponding theorems for spaces of constant curvature

Let now

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{4.1}$$

define the metric of a space of constant positive or negative curvature K. The equation to be solved is still formally the same, namely,

$$g^{\mu\nu}V_{\mu\nu}+\kappa^2 V = 0, \tag{4.2}$$

though no point-transformation will now reduce this to the form $\nabla^2 V+\kappa^2 V=0$.

Let s be the geodesic distance between a fixed point $(\bar{x}^\mu)$ and the variable point (x^μ), and let

$$Q=\cos(K^{\frac{1}{2}}s). \tag{4.3}$$

It will be convenient to employ this function Q rather than the function $\Omega=\frac{1}{2}s^2$ used in the case of a flat space.

As before, let $p_{(\alpha)}$ $(\alpha=1,2,\ldots,n)$ be n functions of a parameter u, subject to the condition

$$\bar{g}^{\mu\nu}p_{(\mu)}p_{(\nu)}=0. \tag{4.4}$$

Define u as a scalar function of the variables (x^μ) by the equation

$$Q^{(\alpha)}p_{(\alpha)}=0, \tag{4.5}$$

where $$Q^{(\alpha)}=\bar{g}^{\alpha\beta}\partial Q/\partial\bar{x}^\beta.$$

Equation (4.5) replaces equation (3.4) of the previous section. Let also

$$\eta=Q^{(\alpha)}p'_{(\alpha)} \tag{4.6}$$

and $$\theta^2=\bar{g}^{\mu\nu}p'_{(\mu)}p'_{(\nu)}. \tag{4.7}$$

Then the theorems to be proved are as follows:

THEOREM C. *A solution of the partial differential equation*

$$g^{\mu\nu}V_{\mu\nu}=0 \tag{4.8}$$

is $$V=f(u), \tag{4.9}$$

where $f(u)$ is an arbitrary function of u.

THEOREM D. *A solution of the partial differential equation*

$$g^{\mu\nu}V_{\mu\nu}+\kappa^2 V=0 \tag{4.10}$$

is $$V=\phi(u)H(a,b;c;\xi), \tag{4.11}$$

where $\phi(u)$ is an arbitrary function of u, and $H(a,b;c;\xi)$ is any convergent solution of the hypergeometric equation

$$\xi(1-\xi)\frac{d^2H}{d\xi^2}+\{c-(a+b+1)\xi\}\frac{dH}{d\xi}-abH=0 \tag{4.12}$$

in which

$$\left.\begin{aligned} a&=\tfrac{1}{4}[n-1+\{(n-1)^2+4\kappa^2K^{-1}\}^{\frac{1}{2}}]\\ b&=\tfrac{1}{4}[n-1-\{(n-1)^2+4\kappa^2K^{-1}\}^{\frac{1}{2}}]\\ c&=\tfrac{3}{2},\qquad \xi=\eta^2\theta^{-2}K^{-1}\end{aligned}\right\}. \tag{4.13}$$

5. Proof of Theorem C

In the following work ordinary subscripts will denote covariant differentiations with respect to the variables (x^μ), while bracketed

suffixes not otherwise defined will represent covariant differentiations with respect to the $\bar{x}$'s, which, though the coordinates of a fixed point, may be regarded as parameters which enter into the discussion. Moreover, the operations of raising and lowering suffixes will be resorted to freely, but it must be remembered that the raising of an ordinary suffix implies contracted multiplication by the coefficients $g^{\lambda\mu}$, while the raising of a bracketed suffix denotes similar multiplication by the constants $\bar{g}^{\lambda\mu}$. Dummy suffixes will sometimes be changed without comment. Since the x's and $\bar{x}$'s are entirely independent of one another, bracketed suffixes representing covariant differentiations may be permuted with unbracketed. As an illustration of these remarks the following should suffice:

$Q_\lambda = \partial Q/\partial x^\lambda$, $\quad Q^\lambda = g^{\lambda\mu}Q_\mu$, $\quad Q_{(\lambda)} = \partial Q/\partial \bar{x}^\lambda$, $\quad Q^{(\lambda)} = \bar{g}^{\lambda\mu}Q_{(\mu)}$,

$Q_{\lambda\mu}$ denotes the second covariant derivative of Q with respect to the x's,

$Q_{(\lambda)(\mu)} = \partial Q_{(\lambda)}/\partial \bar{x}^\mu - \overline{\{\lambda\mu,\alpha\}}Q_{(\alpha)}$, where the Christoffel symbol is evaluated at $(\bar{x}^\mu)$, and

$Q_{\lambda(\mu)} = Q_{(\mu)\lambda} = \partial^2 Q/\partial x^\lambda \partial \bar{x}^\mu$.

We first observe, then, that Q satisfies the identical relations*

$$Q^\lambda Q_\lambda = K(1-Q^2), \tag{5.1}$$

$$Q_{\mu\nu} = -Kg_{\mu\nu}Q, \tag{5.2}$$

whence, raising the suffix ν and contracting,

$$Q^\mu_\mu = -nKQ. \tag{5.3}$$

Differentiating (5.1) with respect to $\bar{x}^\alpha$ and raising the suffix (α),

$$Q^{(\alpha)}_\lambda Q^\lambda = -KQQ^{(\alpha)}, \tag{5.4}$$

since $Q_\lambda Q^{(\alpha)\lambda} = Q^{(\alpha)}_\lambda Q^\lambda$. Differentiating again (covariantly) with respect to $\bar{x}^\beta$ and raising the suffix,

$$Q^{(\alpha)}_\lambda Q^{(\beta)\lambda} + Q^{(\alpha)(\beta)}_\lambda Q^\lambda = -KQ^{(\alpha)}Q^{(\beta)} - KQQ^{(\alpha)(\beta)}. \tag{5.5}$$

Since Q by its definition is unchanged when the x's and $\bar{x}$'s are interchanged, it follows from (5.2) that

$$Q_{(\alpha)(\beta)} = -K\bar{g}_{\alpha\beta}Q,$$

so, raising the suffixes,

$$Q^{(\alpha)(\beta)} = -K\bar{g}^{\alpha\beta}Q, \tag{5.6}$$

whence

$$Q^{(\alpha)(\beta)}_\lambda = -K\bar{g}^{\alpha\beta}Q_\lambda. \tag{5.7}$$

* *Quart. J. of Math.* (Oxford), 1 (1930), 148.

Substituting from (5.6) and (5.7) in (5.5), we deduce that

$$Q^{(\alpha)}_{\lambda}Q^{(\beta)\lambda} = K\bar{g}^{\alpha\beta}Q_{\lambda}Q^{\lambda} - KQ^{(\alpha)}Q^{(\beta)} + K^2\bar{g}^{\alpha\beta}Q^2,$$

whence, by (5.1),

$$Q^{(\alpha)}_{\lambda}Q^{(\beta)\lambda} = K^2\bar{g}^{\alpha\beta} - KQ^{(\alpha)}Q^{(\beta)}. \tag{5.8}$$

Again, differentiating (5.3) with respect to $\bar{x}^{\alpha}$ and raising the suffix,

$$Q^{(\alpha)\mu}_{\ \ \ \ \mu} = -nKQ^{(\alpha)}. \tag{5.9}$$

We are now in a position to begin the proof of Theorem C. Differentiating the given relation (4.4) with respect to u,

$$\bar{g}^{\mu\nu}p'_{(\mu)}p_{(\nu)} = 0. \tag{5.10}$$

We now show that the scalar u defined by (4.5) satisfies the relation

$$u_{\lambda}u^{\lambda} = 0. \tag{5.11}$$

For, differentiating (4.5) with respect to x^{λ},

$$Q^{(\alpha)}p'_{(\alpha)}u_{\lambda} + Q^{(\alpha)}_{\lambda}p_{(\alpha)} = 0,$$

so

$$u_{\lambda}\eta = -Q^{(\alpha)}_{\lambda}p_{(\alpha)}, \tag{5.12}$$

where η is defined by (4.6). Hence

$$u^{\lambda}\eta = -Q^{(\beta)\lambda}p_{(\beta)}, \tag{5.13}$$

whence

$$\begin{aligned} u_{\lambda}u^{\lambda}\eta^2 &= Q^{(\alpha)}_{\lambda}Q^{(\beta)\lambda}p_{(\alpha)}p_{(\beta)} \\ &= K^2\bar{g}^{\alpha\beta}p_{(\alpha)}p_{(\beta)} - KQ^{(\alpha)}Q^{(\beta)}p_{(\alpha)}p_{(\beta)}, \end{aligned}$$

by (5.8). The first term on the right vanishes on account of the condition (4.4) imposed on the arbitrary functions $p_{(\alpha)}$, and the second in virtue of (4.5). Hence the truth of (5.11) is established if we assume, as we must, that the functions $p_{(\alpha)}$ are chosen so that $\eta \neq 0$.

The next problem is to show that

$$u^{\lambda}_{\lambda} = 0, \tag{5.14}$$

where $u^{\lambda}_{\lambda} = g^{\lambda\mu}u_{\lambda\mu}$, the suffixes as usual denoting covariant differentiations.

Differentiating (5.12) covariantly with respect to x^{μ}, raising the suffix and contracting,

$$\begin{aligned} u^{\lambda}_{\lambda}\eta + u_{\lambda}\eta^{\lambda} &= -Q^{(\alpha)\lambda}_{\ \ \ \ \lambda}p_{(\alpha)} - Q^{(\alpha)}_{\lambda}p'_{(\alpha)}u^{\lambda} \\ &= nKQ^{(\alpha)}p_{(\alpha)} + Q^{(\alpha)}_{\lambda}Q^{(\beta)\lambda}p'_{(\alpha)}p_{(\beta)}\eta^{-1} \end{aligned}$$

by (5.9) and (5.13). By (4.5) the first term on the right is zero, and by (5.8),

$$Q^{(\alpha)}_{\lambda}Q^{(\beta)\lambda}p'_{(\alpha)}p_{(\beta)} = K^2\bar{g}^{\alpha\beta}p'_{(\alpha)}p_{(\beta)} - KQ^{(\alpha)}Q^{(\beta)}p'_{(\alpha)}p_{(\beta)} = 0 \tag{5.15}$$

by (5.10) and (4.5).

So
$$u^{\lambda}_{\lambda}\eta + u_{\lambda}\eta^{\lambda} = 0. \tag{5.16}$$
But, by (4.6),
$$\eta^{\lambda} = Q^{(\alpha)\lambda}p'_{(\alpha)} + Q^{(\alpha)}p''_{(\alpha)}u^{\lambda}. \tag{5.17}$$
Hence, using (5.11),
$$u_{\lambda}\eta^{\lambda} = Q^{(\alpha)\lambda}u_{\lambda}p'_{(\alpha)} = -Q^{(\alpha)\lambda}Q^{(\beta)}_{\lambda}p'_{(\alpha)}p_{(\beta)}\eta^{-1}, \text{ by (5.12)},$$
and this vanishes by (5.15). Hence
$$u_{\lambda}\eta^{\lambda} = 0, \tag{5.18}$$
whence $u^{\lambda}_{\lambda} = 0$ by (5.16).

The proof of Theorem C is now immediate. For if $V = f(u)$ is an arbitrary function of u, then
$$V_{\lambda} = f'(u)u_{\lambda}, \qquad V^{\lambda}_{\lambda} = f''(u)u^{\lambda}_{\lambda} + f'(u)u^{\lambda}u_{\lambda} = 0,$$
by (5.11) and (5.14). That is, V satisfies the partial differential equation
$$g^{\lambda\mu}V_{\lambda\mu} = 0.$$

6. Proof of Theorem D

The equation to be solved is
$$V^{\lambda}_{\lambda} + \kappa^2 V = 0. \tag{6.1}$$
The scalar u is defined as a function of the x's by
$$Q^{(\alpha)}p_{(\alpha)} = 0, \tag{6.2}$$
where
$$\bar{g}^{\alpha\beta}p_{(\alpha)}p_{(\beta)} = 0; \tag{6.3}$$
whence, as above,
$$\bar{g}^{\alpha\beta}p'_{(\alpha)}p_{(\beta)} = 0. \tag{6.4}$$
Differentiating the last equation with respect to u, we obtain
$$\bar{g}^{\alpha\beta}p''_{(\alpha)}p_{(\beta)} = -\bar{g}^{\alpha\beta}p'_{(\alpha)}p'_{(\beta)} = -\theta^2, \tag{6.5}$$
where θ is the function introduced above in (4.7). Also, we have already defined
$$\eta = Q^{(\alpha)}p'_{(\alpha)}. \tag{6.6}$$
For the purpose of the discussion it is assumed that the arbitrary functions $p_{(\alpha)}$ are so chosen that neither of the quantities θ, η is zero.

Now assume
$$V = f(u, \eta)$$
as a solution of (6.1), where f is a function of u and η to be determined. Differentiating with respect to x^{λ},
$$V_{\lambda} = f_u u_{\lambda} + f_{\eta}\eta_{\lambda},$$
where the suffixes u and η denote partial differentiations of f with respect to u and η respectively. Differentiating the last equation covariantly with respect to x^{μ}, raising the suffix and contracting,
$$V^{\lambda}_{\lambda} = f_{uu}u_{\lambda}u^{\lambda} + f_u u^{\lambda}_{\lambda} + 2f_{u\eta}u_{\lambda}\eta^{\lambda} + f_{\eta\eta}\eta_{\lambda}\eta^{\lambda} + f_{\eta}\eta^{\lambda}_{\lambda},$$

so by (5.11), (5.14), and (5.18),

$$V^{\lambda}_{\lambda} = f_{\eta\eta}\eta^{\lambda}\eta_{\lambda} + f_{\eta}\eta^{\lambda}_{\lambda}. \tag{6.7}$$

Now multiplying (5.17) by η_{λ} and using (5.18),

$$\eta^{\lambda}\eta_{\lambda} = Q^{(\alpha)\lambda}\eta_{\lambda}p'_{(\alpha)}.$$

But by (5.17),

$$\eta_{\lambda} = Q^{(\beta)}_{\lambda}p'_{(\beta)} + Q^{(\beta)}p''_{(\beta)}u_{\lambda}, \tag{6.8}$$

so

$$\begin{aligned}\eta^{\lambda}\eta_{\lambda} &= Q^{(\alpha)\lambda}Q^{(\beta)}_{\lambda}p'_{(\alpha)}p'_{(\beta)} + Q^{(\alpha)\lambda}Q^{(\beta)}p'_{(\alpha)}p''_{(\beta)}u_{\lambda} \\ &= Q^{(\alpha)\lambda}Q^{(\beta)}_{\lambda}p'_{(\alpha)}p'_{(\beta)} - Q^{(\alpha)\lambda}Q^{(\beta)}Q^{(\gamma)}_{\lambda}p'_{(\alpha)}p''_{(\beta)}p_{(\gamma)}\eta^{-1}, \text{ by (5.12)}, \\ &= Q^{(\alpha)\lambda}Q^{(\beta)}_{\lambda}\{p'_{(\alpha)}p'_{(\beta)} - Q^{(\gamma)}p'_{(\alpha)}p_{(\beta)}p''_{(\gamma)}\eta^{-1}\},\end{aligned}$$

by a change of dummy suffixes in the second term. Putting in the value of $Q^{(\alpha)\lambda}Q^{(\beta)}_{\lambda}$ obtained from (5.8), multiplying out and using (6.2), (6.3), (6.4), (6.5), and (6.6), we get

$$\eta^{\lambda}\eta_{\lambda} = K^2\theta^2 - K\eta^2. \tag{6.9}$$

It is now necessary to evaluate η^{λ}_{λ}. Differentiating (6.8) covariantly with respect to x^{μ}, raising the suffix and contracting,

$$\begin{aligned}\eta^{\lambda}_{\lambda} &= Q^{(\beta)\lambda}_{\ \ \lambda}p'_{(\beta)} + 2Q^{(\beta)}_{\lambda}p''_{(\beta)}u^{\lambda} + Q^{(\beta)}p'''_{(\beta)}u_{\lambda}u^{\lambda} + Q^{(\beta)}p''_{(\beta)}u^{\lambda}_{\lambda} \\ &= -nKQ^{(\beta)}p'_{(\beta)} + 2Q^{(\beta)}_{\lambda}p''_{(\beta)}u^{\lambda},\end{aligned}$$

by (5.9), (5.11), and (5.14). Inserting the value of u^{λ} obtained from (5.13) and using (5.8), we get

$$\begin{aligned}\eta^{\lambda}_{\lambda} &= -nKQ^{(\beta)}p'_{(\beta)} - 2K^2\bar{g}^{\alpha\beta}p_{(\alpha)}p''_{(\beta)}\eta^{-1} + 2KQ^{(\alpha)}Q^{(\beta)}p_{(\alpha)}p''_{(\beta)}\eta^{-1} \\ &= -nK\eta + 2K^2\theta^2\eta^{-1}\end{aligned} \tag{6.10}$$

by (6.6), (6.5), and (6.2). Hence, substituting from (6.9) and (6.10) in (6.7) we deduce that

$$V^{\lambda}_{\lambda} = (K^2\theta^2 - K\eta^2)f_{\eta\eta} + (2K^2\theta^2\eta^{-1} - nK\eta)f_{\eta}. \tag{6.11}$$

Hence, if $V = f(u, \eta)$ satisfies the partial differential equation (6.1), f must satisfy the partial differential equation

$$(K^2\theta^2 - K\eta^2)\frac{\partial^2 f}{\partial\eta^2} + \left(\frac{2K^2\theta^2}{\eta} - nK\eta\right)\frac{\partial f}{\partial\eta} + \kappa^2 f = 0. \tag{6.12}$$

Putting for convenience

$$K\theta^2 = h^2 \qquad \kappa^2K^{-1} = \lambda^2, \tag{6.13}$$

this becomes

$$(h^2 - \eta^2)\frac{\partial^2 f}{\partial\eta^2} + \frac{1}{\eta}(2h^2 - n\eta^2)\frac{\partial f}{\partial\eta} + \lambda^2 f = 0.$$

Since only derivatives with respect to η appear in this equation, u may be treated as a constant in integrating it. Noticing that h is independent of η, introduce the new variable ξ defined by

$$\eta^2 = h^2\xi, \tag{6.14}$$

and (6.12) reduces to

$$\xi(1-\xi)\frac{\partial^2 f}{\partial \xi^2}+\left(\frac{3}{2}-\frac{n+1}{2}\xi\right)\frac{\partial f}{\partial \xi}+\frac{\lambda^2}{4}f=0. \tag{6.15}$$

Writing this in the form

$$\xi(1-\xi)\frac{\partial^2 f}{\partial \xi^2}+\{c-(a+b+1)\xi\}\frac{\partial f}{\partial \xi}-abf=0, \tag{6.16}$$

so that $\quad c=\frac{3}{2}, \quad a+b+1=\frac{1}{2}(n+1), \quad ab=-\frac{1}{4}\lambda^2,$ (6.17)

it is evident that $\quad f=\phi(u)H(a,b;c;\xi),$ (6.18)

where $\phi(u)$ is an arbitrary function of u, and H is any convergent solution of the hypergeometric equation. Putting the values of λ, h given by (6.13) in (6.14) and (6.17), and solving the latter for a and b, the truth of Theorem D becomes obvious.

7. Conclusion

In Theorem D consider in particular the solution

$$V=\phi(u)F(a,b;c;\xi) \tag{7.1}$$

where F is the hypergeometric function, convergent when $|\xi|<1$. Make $K\to 0$, so that the space tends to a flat space. By (4.3),

$$Q=\cos(K^{\frac{1}{2}}s),$$

hence $\quad Q^{(\alpha)}K^{-1}=-\sin(K^{\frac{1}{2}}s)s^{(\alpha)}K^{-\frac{1}{2}}\to -ss^{(\alpha)}=-\Omega^{(\alpha)},$ (7.2)

since by definition $\Omega=\frac{1}{2}s^2$. So dividing (6.2) by K, the relation defining u as a function of the x's becomes

$$\Omega^{(\alpha)}p_{(\alpha)}=0, \tag{7.3}$$

which is the same as (3.4) with $a=0$. Moreover, it is a matter of mere algebra to show that

$$F(a,b;c;\xi)\to(\theta/\kappa\eta)\sin(\kappa\eta/\theta), \tag{7.4}$$

where η now has the meaning it had in § 3, namely, $\eta=-\Omega^{(\alpha)}p'_{(\alpha)}$. So, since θ/κ is a function of u only, it may be absorbed into the arbitrary function $\phi(u)$, and the solution (7.1) thus tends to

$$V=\phi(u)\eta^{-1}\sin(\kappa\eta\theta^{-1}), \tag{7.5}$$

which is obviously in agreement with Theorem B.

Similarly, it can be shown that the complementary solution

$$V=\phi(u)(\xi)^{1-c}F(a-c+1,b-c+1;2-c;\xi) \tag{7.6}$$

tends to the form $\quad V=\chi(u)\eta^{-1}\cos(\kappa\eta\theta^{-1}),$ (7.7)

where $\chi(u)$ is arbitrary.

VORONOÏ'S SUMMATION FORMULA

By J. R. WILTON (*Adelaide*)

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LET $d(n)$ denote the number of divisors of the positive integer n, and let $f(t)$ be a function of bounded variation in (a, b), where $0 < a < b$ For convenience I put

$$f(a-0) = f(b+0) = 0.$$

Then the formula in question (due essentially to Voronoï*) is that,

$$\textit{if} \qquad S_{a,b} = \tfrac{1}{2} \sum_{a \leqslant n \leqslant b} \{f(n+0)+f(n-0)\} d(n) - \int_a^b (\log t + 2\gamma) f(t)\, dt, \quad (1)$$

$$\textit{then} \qquad S_{a,b} = 2\pi \sum_{n=1}^{\infty} d(n) \int_a^b M_0\{4\pi\sqrt{(nt)}\} f(t)\, dt, \quad (1.1)$$

where γ is Euler's constant, and

$$M_0(z) = -Y_0(z) + (2/\pi) K_0(z),$$

Y_0 and K_0 being Bessel functions of the second and third kinds.†

The formula (1.1) has been proved very recently by Dixon and Ferrar‡ under the more restrictive condition that $f(x)$ has a bounded second differential coefficient in (a, b). It is the purpose of this brief note to show that rather more than (1.1) follows very easily from a theorem which I have given still more recently.§

Let $$L_n(z) = -Y_n(z) - (2/\pi) K_n(z),$$
and let $$D(x) = {\sum_{n \leqslant x}}' d(n) = \sum_{n \leqslant x} d(n) - \tfrac{1}{2} d(x),$$
where $d(x) = 0$, if x is not a positive integer. Finally, let

$$\Delta(x) = D(x) - x \log x - (2\gamma - 1)x - \tfrac{1}{4}; \quad (1.2)$$

then the theorem last referred to may be stated as follows:

If $t \geqslant a > 0$, $N > A > 0$, $\epsilon > 0$, then

$$\sum (t, N) = \Delta(t) - {\sum_{n \leqslant N}}' d(n) \sqrt{\left(\frac{t}{n}\right)} L_1\{4\pi\sqrt{(nt)}\} \quad (2)$$

$$= T + O(t^{\frac{1}{4}+\epsilon} N^{-\frac{1}{4}}) + O\{(Nt)^{-\frac{1}{4}} |\log N|\}, \quad (2.1)$$

uniformly in $a \leqslant t \leqslant b$,

* *Annales de l'École Normale* (3) 21 (1904), 516–29.

† Watson, *Theory of Bessel Functions* (1922), pp. 60, 64, and 80.

‡ *Quart. J. of Math.* (Oxford), 2 (1931), 31–54.

§ 'On Dirichlet's divisor problem': *Proc. Royal Soc.* (A) 134 (1931), 192–202, Theorem 3.

where $$T = 0, \quad \text{if } t < \tfrac{1}{2},$$

$$T = \frac{\operatorname{sgn}(t-\tau)}{\pi}\left(\frac{t}{\tau}\right)^{\frac{1}{4}} d(\tau) \int\limits_{4\pi\sqrt{N}|\sqrt{t}-\sqrt{\tau}|}^{\infty} \frac{\sin u}{u}\,du, \text{ if } t \geqslant \tfrac{1}{2},\ \tau = [t+\tfrac{1}{2}]. \tag{2.2}$$

Let A denote a positive absolute constant, not the same at different occurrences; let B denote a positive constant, also different at different occurrences, which depends only on ϵ (> 0); and let

(i) $f(t) = 0$ when $t < a$ and when $t > b$,

(ii) $V_h^k f(t)$ denote the total variation of $f(t)$ over (h, k),

(iii) $V_h^{k-0} f(t) = \lim\limits_{\epsilon \to +0} V_h^{k-\epsilon} f(t)$.

Then, by the method which Dixon and Ferrar use in their § 4.3, I prove

THEOREM 1. *If* $S_{a,b}$ *is given by* (1), *and if* $0 < \theta < \frac{1}{2}$, $N > 1+A$, *then*

$$\left| S_{a,b} - 2\pi \sum_{n \leqslant N}{}' d(n) \int\limits_a^b M_0\{4\pi\sqrt{(nt)}\} f(t)\,dt \right|$$
$$< \left\{ \frac{Bb^{\frac{1}{4}+\epsilon}}{\theta\sqrt{N}} + \frac{A \log N}{(Na)^{\frac{1}{4}}} \right\} \{|f(b)| + V_a^b f(t)\} +$$
$$+ Bb^{\epsilon} \sum_{a \leqslant n \leqslant b} \{V_{n-\theta}^{n-0} f(t) + V_{n+0}^{n+\theta} f(t)\}. \tag{3}$$

In particular, if we put $N = \theta^{-2-A}$, and then make $\theta \to 0$, we obtain (1.1).

There is an analogous theorem in which $d(n)$ is replaced by $r(n)$, the number of integral solutions (positive, negative, and zero) of the Diophantine equation
$$x^2 + y^2 = n,$$
and it may be proved by a precisely similar method.* Let

$$P(x) = \sum_{0 \leqslant n \leqslant x}{}' r(n) - \pi x,$$

$$\textstyle\sum_{x,X} = P(x) - \displaystyle\sum_{1 \leqslant n \leqslant x}{}' r(n) \sqrt{\left(\frac{x}{n}\right)} J_1\{2\pi\sqrt{(nx)}\};$$

then $\sum_{x,X}$ satisfies an inequality analogous to (2.1), and we have

THEOREM 2. *If* $b > a > 0$, $0 < \theta < \frac{1}{2}$, *and* $X > A$, *then*

$$\left| \tfrac{1}{2} \sum_{a \leqslant n \leqslant b} \{f(n+0) + f(n-0)\} r(n) - \pi \sum_{n=0}^{X}{}' r(n) \int\limits_a^b J_0\{2\pi\sqrt{(nx)}\} f(x)\,dx \right|$$
$$< \left\{ \frac{Bb^{\frac{1}{4}+\epsilon}}{\theta\sqrt{X}} + \frac{A}{(aX)^{\frac{1}{4}}} \right\} \{|f(b)| + V_a^b f(t)\} + Bb^{\epsilon} \sum_{a \leqslant n \leqslant b} \{V_{n-\theta}^{n-0} f(t) + V_{n+0}^{n+\theta} f(t)\}.$$

* Cf. Landau, *Vorlesungen über Zahlentheorie* (1927), 2, 274.

It will be sufficient to write out the proof of Theorem 1. From the definition of $D(t)$ we have

$$\tfrac{1}{2}\sum_{a\leqslant n\leqslant b}\{f(n+0)+f(n-0)\}d(n)=\int\limits_a^b f(t)\,dD(t),$$

the integral being a Lebesgue-Stieltjes integral; and therefore, by (1.2),

$$S_{a,b}=\int\limits_a^b f(t)\,d\Delta(t),$$

and therefore, by (2),

$$S_{a,b}-2\pi\sum_{n=1}^{N}{}'\,d(n)\int\limits_a^b M_0\{4\pi\sqrt{(nt)}\}f(t)\,dt$$

$$=\int\limits_a^b f(t)\,d\sum(t,N)$$

$$=f(b)\sum(b,N)-f(a)\sum(a,N)-\int\limits_a^b\sum(t,N)\,df(t).$$

By (2.1) we have

$$\left|\int\limits_a^b\sum(t,N)\,df(t)-\int\limits_a^b T\,df(t)\right|$$

$$<\{Bb^{\frac{1}{2}+\epsilon}N^{-\frac{1}{2}}+A(Na)^{-\frac{1}{4}}\log N\}V_a^b f(t).$$

Hence we have only to approximate to the integral of T. We note first that, in (2.2), $\tau\geqslant 1$, $t<\tau+\frac{1}{2}$; therefore

$$(t/\tau)^{\frac{3}{2}}<(3/2)^{\frac{3}{2}}<2;\ d(\tau)<Bb^{\epsilon};\ \operatorname{sgn}(t-\tau)=0 \text{ when } t=\tau.$$

The modulus of the integral in (2.2) never exceeds $\frac{1}{2}\pi$, and, if $|\tau-t|\geqslant\theta$, it is not greater than

$$(2\pi\surd N\,|\surd t-\surd\tau|)^{-1}\leqslant\surd b/(\pi\theta\surd N).$$

Hence

$$\left|\int\limits_a^b T\,df(t)\right|=\left|\sum_{a\leqslant\tau\leqslant b}\left(\int\limits_{\tau-\frac{1}{2}}^{\tau-\theta}+\int\limits_{\tau+\theta}^{\tau+\frac{1}{2}}+\int\limits_{\tau-\theta}^{\tau-0}+\int\limits_{\tau+0}^{\tau+\theta}\right)T\,df(t)\right|$$

$$<\frac{Bb^{\frac{1}{2}+\epsilon}}{\theta\surd N}V_a^b f(t)+Bb^{\epsilon}\sum_{a\leqslant\tau\leqslant b}\{V_{\tau-\theta}^{\tau-0}f(t)+V_{\tau+0}^{\tau+\theta}f(t)\}.$$

This completes the proof of Theorem 1.

It is easy to state a set of sufficient conditions under which we may extend the interval (a,b) in (1.1) to (a,∞). For example:

If (i) $0 < a < t_0 < \infty$, (ii) $f(t)$ *is of bounded variation in* (a, t_0), (iii) $f(t) \to 0$ *as* $t \to \infty$, (iv) $f(t)$ *is the indefinite integral of* $f'(t)$ *in* $t \geqslant t_0$, *and* (v) *if*

$$\int_{t_0}^{\infty} t^{\frac{1}{4}+\kappa} |f'(t)|\, dt$$

converges for some $\kappa > 0$, *then**

$$S_{a,\infty} = \lim_{b\to\infty} S_{a,b} = 2\pi \sum_{n=1}^{\infty} d(n) \int_a^{\infty} M_0\{4\pi\sqrt{(nt)}\} f(t)\, dt. \qquad (3.1)$$

It is somewhat more difficult to prove the formula corresponding to (1.1) when $a = 0$, and it appears to be necessary to impose a further condition on $f(t)$ as $t \to +0$. Since the formula is additive, it will be sufficient to consider the series of integrals over $(0, a)$, where $0 < a < \frac{1}{2}$ (and therefore $T = 0$). In this case we have

THEOREM 3. *If* (i) $0 < a < \frac{1}{2}$, (ii) $f(t)$ *is of bounded variation in* $(0, a)$, (iii) $0 < \theta < 1$, *and* (iv) $N^\theta a > 1$, *then*

$$\left| \tfrac{1}{4} f(+0) + \int_0^a (\log t + 2\gamma) f(t)\, dt + 2\pi \sum_{n\leqslant N}{}' d(n) \int_0^a M_0\{4\pi\sqrt{(nt)}\} f(t)\, dt \right|$$
$$< AN^{-\frac{1}{4}(1-\theta)} \log N\{|f(a)| + V_0^a f(t)\} + A \log N\, V_{+0}^{N^{-\theta}} f(t). \qquad (4)$$

If the further condition

$$\lim_{x\to+0} \log x\, V_{+0}^{x} f(t) = 0 \qquad (4.1)$$

is imposed upon $f(t)$, *then*

$$\tfrac{1}{4} f(+0) + \int_0^a (\log t + 2\gamma) f(t)\, dt + 2\pi \sum_{n=1}^{\infty} d(n) \int_0^a M_0\{4\pi\sqrt{(nt)}\} f(t)\, dt = 0. \qquad (4.2)$$

The last part of the theorem is an immediate consequence of (4) and (4.1), if we make $N \to \infty$.

The formula (4.2) has been proved by Dixon and Ferrar† when $f''(t)$ is bounded in (δ, a) and $t^{\frac{3}{4}} |f''(t)|$ is integrable over $(0, \delta)$, where $0 < \delta < a$. Dixon and Ferrar insert in (4.2) a term

$$-\tfrac{1}{4} \lim_{t\to+0} t f'(t);$$

but this is evidently zero, for if $0 < t < a$, we have‡

$$\lambda f(t) = \lambda f(\tfrac{1}{2}t) + \tfrac{1}{2}\lambda t f'(\lambda t) \qquad (\tfrac{1}{2} < \lambda < 1).$$

* Of course it does not follow that either series or integral in (1) converges as $b \to \infty$, but if either does, the other does.

† Loc. cit., p. 54, (7.45).

‡ A slight elaboration of the same argument, applied to $f'(t)$, shows that

$$\lim t f'(t) = l \neq 0$$

implies the divergence of the integral over $(0, \delta)$ of $t f''(t)$, and therefore *a fortiori* of $t^{\frac{3}{4}} |f''(t)|$. If is, of course, assumed that the limit exists.

In the proof of Theorem 3, I use, instead of (2.1), the exact expression from which this is derived, namely,*

$$\sum(t,N) = \frac{\log N+2\gamma}{2\pi} L_0\{4\pi\surd(Nt)\} + \frac{1}{2\pi}\int\limits_N^\infty L_0\{4\pi\surd(xt)\}\frac{dx}{x} -$$

$$-\Delta(N)\surd(t/N)L_1\{4\pi\surd(Nt)\} + \chi(t,N), \tag{5}$$

where
$$\chi(t,N) = 2\pi\int\limits_N^\infty \frac{t}{x} L_2\{4\pi\surd(xt)\}\Delta(x)\,dx. \tag{5.1}$$

If $0 < t \leqslant a$, $Nt > A$, then†

$$|\chi(t,N)| < At^{\frac{3}{4}}N^{-\frac{1}{2}} < AN^{-\frac{1}{2}}. \tag{5.2}$$

I also require an inequality for $\chi(t,N)$ when $t < N^{-1}$.

By (5.2) $$|\chi(t,t^{-1})| < At^{\frac{5}{4}} < AN^{-\frac{1}{2}}.$$

There remains for consideration the integral

$$\int\limits_N^{1/t} \frac{t}{x} L_2\{4\pi\surd(xt)\}\Delta(x)\,dx \qquad (Nt < 1),$$

of which the modulus is less than

$$At\int\limits_1^{1/t} \frac{|\Delta(x)|}{x}\,dx < At^{\frac{1}{2}} < AN^{-\frac{1}{2}},$$

if we use $|\Delta(x)| < Ax^{\frac{1}{2}}$. Hence, if $0 \leqslant t \leqslant a$, then

$$|\chi(t,N)| < AN^{-\frac{1}{2}}, \tag{5.3}$$

and (5.3) may be improved by using known, but less elementary, inequalities satisfied by $\Delta(x)$.

I proceed now to the proof of Theorem 3. Since $t < \frac{1}{2}$, $\Delta(t)$ is the indefinite integral of $-(\log t + 2\gamma)$; hence, by (5),

$$-\int\limits_0^a (\log t + 2\gamma)f(t)\,dt = \int\limits_0^a f(t)\,d\Delta(t)$$

$$= 2\pi\sum_{n\leqslant N}{}' d(n)\int\limits_0^a M_0\{4\pi\surd(nt)\}f(t)\,dt - \tag{6}$$

* Wilton, loc. cit., Theorem 1.

† Wilton, loc. cit., (4.24–6). Since $t < \frac{1}{2}$, both (4.24) and (4.25) are null, and (4.26) is $< AN^{-\frac{1}{2}}t^{\frac{3}{4}}\sum n^{-\frac{5}{4}}\,d(n)$.

$$-2\pi\Delta(N)\int_0^a M_0\{4\pi\sqrt{(Nt)}\}f(t)\,dt - \tag{6.1}$$

$$-(\log N+2\gamma)\int_0^a \sqrt{\left(\frac{N}{t}\right)}\, L_1\{4\pi\sqrt{(Nt)}\}f(t)\,dt - \tag{6.2}$$

$$-\frac{1}{2\pi}\int_0^a L_0\{4\pi\sqrt{(Nt)}\}f(t)\,\frac{dt}{t} + \tag{6.3}$$

$$+f(a)\chi(a,N)-\int_0^a \chi(t,N)\,df(t). \tag{6.4}$$

By (5.3) the modulus of (6.4) is less than

$$AN^{-\frac{1}{4}}\{|f(a)|+V_0^a f(t)\}. \tag{7.1}$$

In (6.1) and (6.2) we may replace $f(t)$ by $f(t)-f(+0)$, with an error less than

$$A(Na)^{-\frac{1}{4}}\log N. \tag{7.2}$$

When we make the same change in (6.3), we must add

$$-\frac{f(+0)}{\pi}\int_0^\infty L_0(u)\,\frac{du}{u} = \tfrac{1}{4}f(+0), \tag{7.3}$$

if we wish to retain the same error term (7.2).

In each of the three integrals, thus modified, divide the interval $(0,a)$ into the two parts $(0,N^{-\theta})$ and $(N^{-\theta},a)$, where $0<\theta<1$. In each of the six sub-intervals I employ the second theorem of the mean, in the form

$$\left|\int_b^c f(t)\phi(t)\,dt - f(c)\int_b^c \phi(t)\,dt\right| \leqslant V_b^c f(t) \max_{b\leqslant\alpha<\beta\leqslant c}\left|\int_\alpha^\beta \phi(t)\,dt\right|.$$

The sum of the three integrals over $(N^{-\theta},a)$ is thus seen to introduce an error less than

$$AN^{-\frac{1}{4}(1-\theta)}\log N\{|f(a)|+V_0^a f(t)\}. \tag{7.4}$$

The three integrals over $(0,N^{-\theta})$ are dominated by (6.2); in addition to (7.4) we have, on account of (6.2), to introduce an error term less than

$$A\log N\,.\,V_{+0}^{N^{-\theta}} f(t). \tag{7.5}$$

Since the errors (7.1) and (7.2) are both included in (7.4), Theorem 3 is a consequence of (6) and (7.3)–(7.5).

The corresponding theorem with respect to $r(n)$ is simpler. It depends upon Hardy's formula,*

$$P(x) - \sum_{n=1}^{X}{}' r(n)\sqrt{\left(\frac{x}{n}\right)} J_1\{2\pi\sqrt{(nx)}\} = J_0\{2\pi\sqrt{(Xx)}\} -$$

$$- P(X)\sqrt{\left(\frac{x}{X}\right)} J_1\{2\pi\sqrt{(Xx)}\} + \pi \int_X^\infty \frac{x}{u} J_2\{2\pi\sqrt{(xu)}\} P(u)\, du, \qquad (8)$$

where $x > 0$, $X > 0$; and it may be stated in the form:

THEOREM 4. *If* $0 < \theta < a < \frac{1}{2}$, *$f(t)$ is of bounded variation in* $(0, a)$, *and* $X > A$, *then*

$$\left| f(+0) - \pi \sum_{n=0}^{X}{}' r(n) \int_0^a J_0\{2\pi\sqrt{(nx)}\} f(x)\, dx \right|$$

$$< A(X\theta)^{-\frac{1}{4}}\{|f(a)| + V_0^a f(t)\} + A V_{+0}^{\theta} f(t).$$

The formula obtained by putting $X = \theta^{-1-\Delta}$ and then making $\theta \to 0$ is due to Landau.†

* *Proc. Royal Soc.* (A), 120 (1925), 623–35, a modification of Theorem 1.

† Loc. cit., p. 274.

CONVEX REGIONS IN THE GEOMETRY OF PATHS

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1. Introduction. A classical theorem in differential geometry asserts the existence of a region C_q containing a given point q in a Riemannian space, such that any point in C_q can be joined to q by one and only one geodetic segment which does not leave C_q. A similar theorem holds for the geometry of paths, and is equivalent to the statement that a normal coordinate-system exists having q as origin. There does not, however, seem to be a proof of the theorem that a region exists in which two points are joined by one, and only one, segment of a path which does not leave the region. Such a region will be called *simple*, because not more than one, and *convex*, because at least one path joins any two points. We shall show that any non-singular point in an affine, or projective, space of paths is contained in a simple, convex region which can be made as small as we please.

Instead of the usual 'point-direction' or 'initial conditions' existence theorem for the differential equations to the paths, we use Picard's 'two-point' or 'boundary value' existence theorem. By this means the theorem is proved as a generalization of the remark that the points in a flat affine space, given in cartesian coordinates by

$$V(y^1,\dots,y^n) \leqslant 0,$$

constitute a convex region, if the quadratic form

$$\frac{\partial^2 V}{\partial y^j \partial y^k} dy^j dy^k$$

is positive definite at each point of the hypersurface

$$V(y^1,\dots,y^n) = 0.$$

Unless otherwise stated, an open region will mean an open region in the arithmetic or number space of n dimensions. That is, a set X containing the cell

$$|x^i - x_0^i| < \delta \qquad (i = 1,\dots,n),$$

for some positive δ, where x_0 is any point in X. A *closed region* $\bar{X}$ will mean the closure of an open region, X. The word region used

by itself may mean either an open or a closed region. We deal only with real variables and real functions.

2. An existence theorem. There is a theorem due to E. Picard,* which asserts that differential equations of the form

$$\frac{d^2x^i}{ds^2} = f^i\left(s, x, \frac{dx}{ds}\right) \qquad (i = 1,..., n),$$

admit a unique set of solutions

$$\psi^1(x_0, x_1, s_0, s_1, s),..., \psi^n(x_0, x_1, s_0, s_1, s),$$

satisfying the boundary conditions

$$\left.\begin{aligned} \psi^i(x_0, x_1, s_0, s_1, s_0) &= x_0^i, \\ \psi^i(x_0, x_1, s_0, s_1, s_1) &= x_1^i, \end{aligned}\right\},$$

provided $f^i(s, x, \xi)$ satisfy certain continuity conditions, and s_0, s_1, x_0, x_1 are properly chosen.

We shall have to do with differential equations of the form†

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{2.1}$$

where Γ^i_{jk} are functions of $x^1,..., x^n$. We assume Γ to be defined in the region

$$|x^i| < 2, \tag{2.2}$$

to be bounded, continuous, and to satisfy a Lipschitz condition

$$|\Gamma^i_{jk}(x_1) - \Gamma^i_{jk}(x_0)| \leqslant \Delta \sum_l |x_1^l - x_0^l|, \tag{2.3}$$

x_0 and x_1 being any points in (2.2), and Δ some positive constant. Equations of the form (2.1) have the property that if

$$\psi^i(s) \qquad (s_0 \leqslant s \leqslant s_1),$$

are solutions, so are

$$\phi^i(t) = \psi^i\left\{\frac{s_1 - s_0}{t_1 - t_0}(t - t_0) + s_0\right\} \qquad (t_0 \leqslant t \leqslant t_1),$$

where t_0 and t_1 are constants, arbitrary except that $t_0 \neq t_1$. If λ^i is the maximum of $|\psi'^i(s)|$ as s varies from s_0 to s_1, it follows that

$$\mu^i = \frac{s_1 - s_0}{t_1 - t_0}\lambda^i$$

is the maximum of $|\phi'^i(t)|$ as t varies from t_0 to t_1. This means that there is a certain homogeneity relation between the upper bound

* *Traité d'analyse*, 2nd edition, Paris, 1908, vol. iii, pp. 90–6.

† This note only applies to the restricted geometry of paths as apart from more general theories in which Γ^i_{jk} depend on the direction dx. See J. Douglas, *Annals of Math.* 29 (1928), 143–68.

which must be put on s_1-s_0, and the upper bound which must be put on $|\psi'^i|$. This relation, and the fact that the functions

$$\Gamma^i_{jk}\xi^j\xi^k \tag{2.4}$$

are defined for all values of ξ, are the special features of the equations (2.1), which enable us to prove our theorem.

We first restrict the variables ξ in (2.4) by the conditions

$$|\xi^i| < \lambda, \tag{2.5}$$

where λ is any positive constant. Let M/n^2 be an upper bound for the functions Γ^i_{jk} as x varies in (2.2). Then

$$|\Gamma^i_{jk}\xi^j\xi^k| < M\lambda^2,$$

for values of x in (2.2) and for ξ in (2.5). Further, let

$$\alpha = n^2\Delta, \tag{2.61}$$

$$\beta = 2M/n, \tag{2.62}$$

where Δ is the constant in (2.3).

Then for x_0 and x_1 in (2.2), and ξ_0 and ξ_1 in (2.5), we have

$$|\Gamma^i_{jk}(x_1)\xi^j_1\xi^k_1-\Gamma^i_{jk}(x_0)\xi^j_0\xi^k_0|$$
$$\leqslant |\{\Gamma^i_{jk}(x_1)-\Gamma^i_{jk}(x_0)\}\xi^j_1\xi^k_1|+|\Gamma^i_{jk}(x_0)\xi^j_1\xi^k_1-\Gamma^i_{jk}(x_0)\xi^j_0\xi^k_0|.$$

By (2.3) and (2.61) applied to the first term on the right-hand side, and by the first mean-value* theorem and (2.62), applied to the second term, we have

$$|\Gamma^i_{jk}(x_1)\xi^j_1\xi^k_1-\Gamma^i_{jk}(x_0)\xi^j_0\xi^k_0| \leqslant \lambda^2\alpha\sum_j |x^j_1-x^j_0|+\lambda\beta\sum|\xi^j_1-\xi^j_0|. \tag{2.7}$$

Now let x_0 be any point in the closed region

$$|x^i| \leqslant 1.$$

By Picard's existence theorem, there exists one, and only one, set of solutions to (2.1)

$$\psi^i(x_0, x_1, s_0, s_1, s), \qquad (s_0 \leqslant s \leqslant s_1), \tag{2.8}$$

such that

$$\left.\begin{aligned}\psi^i(x_0,\ldots,s_0) &= x^i_0\\ \psi^i(x_0,\ldots,s_1) &= x^i_1\end{aligned}\right\}, \tag{2.9}$$

provided

$$\left.\begin{aligned}&\frac{M\lambda^2(s_1-s_0)^2}{8}+|x^i_1-x^i_0| < 1\\ &\frac{M\lambda^2(s_1-s_0)}{2}+\frac{|x^i_1-x^i_0|}{(s_1-s_0)} < \lambda\\ &\frac{\theta\lambda^2(s_1-s_0)^2}{8}+\frac{\theta\lambda(s_1-s_0)}{2} < 1\end{aligned}\right\}, \tag{2.10}$$

* $|\partial\Gamma^i_{jk}\xi^j\xi^k/\partial\xi^l| = 2|\Gamma^i_{jl}\xi^j| < 2M\lambda/n.$

where θ is any number such that

$$n\alpha < \theta, \qquad n\beta < \theta.$$

As s varies from s_0 to s_1, $|\psi^i| < 2$ and $|\psi'^i| < \lambda$.

Let
$$\mu = \lambda(s_1 - s_0). \tag{2.11}$$

Then (2.10) may be written

$$M\mu^2/8 + |x_1^i - x_0^i| < 1, \tag{2.121}$$
$$M\mu^2/2 + |x_1^i - x_0^i| < \mu, \tag{2.122}$$
$$\mu^2/8 + \mu/2 < 1/\theta. \tag{2.123}$$

Let
$$|x_1^i - x_0^i| < a,$$

for some positive a less than unity. Then (2.121) and (2.123) are both satisfied for $\mu = 0$. Therefore they are satisfied for every small μ, and for every x_0, x_1 such that

$$|x_0^i| < 1, \qquad |x_1^i - x_0^i| < a. \tag{2.13}$$

Therefore we can find μ_0 such that (2.121) and (2.123) are satisfied, subject to (2.13), for $\mu \leqslant \mu_0$, and such that

$$M\mu_0^2/2 < \mu_0.$$

Then (2.12) are all satisfied for $\mu \leqslant \mu_0$, and

$$|x_1^i - x_0^i| \leqslant 2\delta',$$

for any positive δ' such that

$$\left.\begin{aligned} 2\delta' &< a \\ 2\delta' &< \mu_0 - M\mu_0^2/2 \end{aligned}\right\}.$$

In (2.5) let $\lambda = \mu_0$. Then there is one, and only one, solution (2.8), where $s_0 = 0$, $s_1 = 1$, and x_0 and x_1 are in the closed region

$$|x^i| \leqslant \delta'. \tag{2.14}$$

A set of n functions $\psi^i(x_0, x_1, s)$ of $2n+1$ variables, x_0, x_1, and s, is thus defined for x_0 and x_1 in (2.14) and for

$$0 \leqslant s \leqslant 1.$$

The functions $\psi^i(x_0, x_1, s)$ are continuous in the $2n+1$ variables.

On the assumption that bounded, continuous derivatives $\partial\Gamma/\partial x$ exist, this is a consequence of a general theorem proved by G. A. Bliss.* According to this theorem the functions $\psi^i(x_0, x_1, s)$ are

* *Trans. American Math. Soc.* 5 (1904), 113–25, in particular pp. 114–16. Bliss was considering the one-dimensional case, and the extension of his result from $n = 1$ to any n involves a modification of the last paragraph on p. 115. He proved that a certain derivative $\dfrac{\partial\phi}{\partial\eta_0'}$ does not vanish. For $n > 1$ it is

differentiable. We shall only need their continuity, and this follows from the Lipschitz condition, without assuming the existence* of $\partial\Gamma/\partial x$.

This section may be summarized as follows. Any integral curve of (2.1) given by

$$x^i = \psi^i(x_0, x_1, s), \qquad (s_0 \leqslant s \leqslant s_1), \tag{2.15}$$

where ψ^i satisfy (2.9), is called a path,† and x_0 and x_1 its end-points. Any parameter referred to which a path satisfies (2.1) is called an affine parameter, and the class of affine parameters consists of those, and only those, related to a given one by linear equations

$$t = \alpha s + \beta \qquad (\alpha \neq 0).$$

On each path we define a function

$$\mu(s) = \lambda(s)(s - s_0) \qquad (s_0 \leqslant s \leqslant s_1), \tag{2.16}$$

where $\lambda(s)$ is the maximum of

$$|\psi'^1(x_0, x_1, \sigma)|, \ldots, |\psi'^n(x_0, x_1, \sigma)|,$$

when σ varies from s_0 to s. The function $\mu(s)$ is continuous and strictly monotonic in s. Though it is not invariant under transformations of coordinates $x \to y$, it is invariant under transformations from one affine parameter to another. The path (2.15) will be described as a μ_0-path if

$$\mu(s) \leqslant \mu_0,$$

as s varies from s_0 to s_1. We have shown that:

One and only one μ_0-path has as its end-points a given pair of points in the closed region (2.14), *and the path varies continuously with the end-points.*

necessary to show that a corresponding Jacobian $\left|\dfrac{\partial\phi^i}{\partial\eta_0'^j}\right|$ does not vanish. This can be done by using the same argument as in the one-dimensional case to show that $\dfrac{\partial\phi^i}{\partial\eta_0'^j}\lambda^j \neq 0$, where $\lambda^1, \ldots, \lambda^n$ are given constants not all zero. Otherwise the same arguments apply for $n > 1$ as for $n = 1$.

* Let $\psi^i(x_0, x_1, s)$ be solutions to $\dfrac{d^2x^i}{ds^2} = f^i(s, x, dx/ds)$, which take on the boundary values x_0 and x_1 for $s = 0$ and $s = b$ respectively. On the assumption that $f^i(s, x, \xi)$ satisfy a Lipschitz condition, the continuity of $\psi^i(x_0, x_1, s)$, in x_0 and x_1, follows by an argument similar to that used by Picard (loc. cit., p. 93) in proving the convergence of approximations to a solution. We do not give this argument because it is quite straightforward, and because the existence of $\partial\Gamma/\partial x$ is necessary to so many theorems in the geometry of paths.

† We describe as a path what would usually be called a segment of a path, including the end-points.

In particular, if $x_0 = x_1$, the only μ_0-path both of whose end-points coincide with x_0 is the 'degenerate' path

$$x^i = x_0^i.$$

Thus a non-degenerate μ_0-path cannot be closed, nor can it have a double point.

3. Simple regions. We shall now show that a positive δ exists, such that not more than one path joins a given pair of points x_0 and x_1 in the region $\bar{X}_\delta$, given by

$$|x^i| \leqslant \delta, \tag{3.1}$$

without leaving $\bar{X}_\delta$. A region having this property will be described as *simple*. Any region, open or closed which is contained in a *simple* region is obviously simple.

Since not more than one μ_0-path joins a given pair of points in (2.14), it will be sufficient to prove the following:

There exists a positive $\delta \leqslant \delta'$, such that any path $\gamma(x_0, x_1)$, joining a given pair of points in $\bar{X}_\delta$ which is not a μ_0-path contains at least one point outside $\bar{X}_\delta$.

Let $$x^i = \psi^i(x_0, x_1, s) \qquad (0 \leqslant s \leqslant 1), \tag{3.2}$$
be any path joining x_0 to x_1, where x_0 and x_1 are in (2.14). From (2.1) we have
$$|\psi''^i(s)| \leqslant M\lambda(s)^2,$$
and therefore*
$$|\psi(x_0, x_1, s) - x_0| \geqslant \mu(s) - \tfrac{1}{2}M\mu(s)^2, \tag{3.3}$$

* Let $f(x)$ be any function defined for $0 \leqslant x \leqslant 1$, whose derivatives $f'(x)$ and $f''(x)$ exist and are continuous in this interval. Let
$$M_1(x) = \max|f'(\xi)|, \qquad M_2(x) = \max|f''(\xi)|,$$
as ξ varies from 0 to x. For a given x between 0 and 1 there is an x_0 such that
$$|f'(x_0)| = M_1(x) \qquad (0 \leqslant x_0 \leqslant x),$$
and by the first mean-value theorem we have
$$f'(x_0) - f'(\xi) \leqslant M_2(x)|x_0 - \xi|,$$
for any ξ between 0 and x. That is to say,
$$f'(\xi) \geqslant f'(x_0) - M_2(x)|x_0 - \xi|,$$
and on integrating both sides from 0 to x and simplifying, we have
$$f(x) - f(0) \geqslant f'(x_0)x - \tfrac{1}{2}M_2(x)x^2.$$
Therefore $$|f(x) - f(0)| \geqslant M_1(x)x - \tfrac{1}{2}M_2(x)x^2,$$
which is the result used in the text. Of course we strengthen the inequality if we replace $M_2(x)$ by a greater function.

where the omission of indices means 'at least one of $|\psi^1-x_0^1|,...,$ $|\psi^n-x_0^n|$ exceeds the expression on the right-hand side'.

Now $\mu(0) = 0$ and $t-\frac{1}{2}Mt^2$ increases steadily as t increases from 0 to $1/M$. Let r be any number less than μ_0 and less than $1/M$. If (3.2) is not a μ_0-path,

$$r < \mu(s) < 1/M$$

for some value of s between 0 and 1. For this value of s we have, from (3.3),

$$|\psi(x_0, x_1, s)-x_0| > 2\delta \tag{3.4}$$

for any δ such that $\quad 0 < 2\delta \leqslant r-\frac{1}{2}Mr^2.$

It follows that any path which is not a μ_0-path has at least one point outside the closed region $\bar{X}_\delta$, given by

$$|x^i| \leqslant \delta. \tag{3.7}$$

As explained above, if we take $\delta \leqslant \delta'$ the region $\bar{X}_\delta$ is simple.

4. Convex regions. A region X open or closed, will be called *convex* if any two points in X are joined by at least one path which does not leave X. We may express this by saying that any two points in X are *visible** from each other in X.

Let X be any open region and $\bar{X}$ its closure. Then a path in $\bar{X}$ will be described as 'in X' if it is contained in X, with the possible exception of either end-point, or both. Two points in $\bar{X}$ will be described as visible from each other in X if they are joined by a path in X, and $\bar{X}$ will be described as *completely convex* if any two points in $\bar{X}$ are visible from each other in X.

Let

$$V(x) = 0$$

be the equation of a closed hypersurface V which lies entirely in the closed region $\bar{X}_\delta$, given by (3.7). Let V have the properties:

1. The closed region, $\bar{C}$, given by

$$V(x) \leqslant 0$$

is connected.†

* A term suggested by K. Menger, *Math. Annalen*, 100 (1928), 81. We may think of X as filled with substance which conducts light along paths, all the space except X being opaque.

† That is to say, any two points in $\bar{C}$ can be joined by a continuous curve which does not leave $\bar{C}$.

2. The quadratic form

$$V_{,jk}\,dx^j dx^k = \left(\frac{\partial^2 V}{\partial x^j \partial x^k} - \frac{\partial V}{\partial x^i}\,\Gamma^i_{jk}\right) dx^j dx^k$$

is positive definite at each point x on* V.

Since the closed region $\bar{C}$, consisting of points in and on V, is contained in the simple region $\bar{X}_\delta$, it is simple, and we shall show that it is completely convex.

Let x_0 be any point on V, and

$$x^i = x^i(s) \tag{4.1}$$

a path which touches V at x_0, that is to say,

$$V(x_0) = 0, \qquad \left(\frac{\partial V}{\partial x^j}\right)_{x_0} \xi^j_0 = 0,$$

where $x(s_0) = x_0$, $\xi_0 = \left(\frac{dx}{ds}\right)_0$. Since $x^i(s)$ satisfy (2.1), it follows that

$$V\{x(s+\Delta s)\} = V_{,jk}\,\xi^j_0 \xi^k_0\,\Delta s^2 + \ldots, \tag{4.2}$$

and therefore $V\{x(s_0+\Delta s)\}$ is positive for small values of Δs. That is to say, all points on a tangent path to V, which are near the point of contact, lie outside V.

Let a and b be any two points in $\bar{C}$. Either the point pair (a, b) is visible,† or else the path joining a to b contains at least one point which is outside V. For if a and b are invisible the path ab has at least one inner point x on V, and if ab contains no point outside V it touches V at x. But the possibility of tangency from the inside is excluded by the second condition on V.

It follows that the totality of visible point pairs in V is a closed set in the $2n$-dimensional region

$$|x^i| \leqslant \delta, \qquad |y^i| \leqslant \delta,$$

* For a sufficiently small positive r such a hypersurface is given parametrically by

$$(a).\; x^i = \bar{x}^i - \tfrac{1}{2}(\Gamma^i_{jk})_0 \bar{x}^j \bar{x}^k,$$
$$(b)\; \sum \bar{x}^i \bar{x}^i - r^2 = 0.$$

For the equations (a) define a transformation to a coordinate-system $\bar{x}$ in which the components, $\bar{\Gamma}^i_{jk}$, of the affine connexion vanish at the origin, and in which the equation to V is (b). In the coordinates $\bar{x}$ we have

$$\bar{V}_{,jk}\,d\bar{x}^j d\bar{x}^k = (\delta_{jk} - \sum_i \bar{x}^i \bar{\Gamma}^i_{jk})\,d\bar{x}^j\,d\bar{x}^k,$$

and for all sufficiently small values of r this quadratic form is positive definite at points on the hypersurface V.

† A pair of points in $\bar{C}$ will be described as visible, if one is visible from the other in the open region C, and invisible otherwise.

which is the product of $\bar{X}_\delta$ with itself. For let (a, b) be any point pair on the boundary of the visible point pairs, and let

$$(a_1, b_1), \qquad (a_2, b_2), \ldots, \qquad (a_\alpha, b_\alpha), \ldots$$

be a sequence of visible point pairs converging to (a, b). Each of the paths

$$x^i = \psi^i(a_\alpha, b_\alpha, s)$$

lies in C, by the definition of visibility. On each of the paths a_α, b_α, let the parameter s be chosen so that

$$\left.\begin{aligned} \psi^i(a_\alpha, b_\alpha, 0) &= a^i_\alpha \\ \psi^i(a_\alpha, b_\alpha, 1) &= b^i_\alpha \end{aligned}\right\}.$$

Let the parameter on the path ab be similarly chosen. Then for each value of s between 0 and 1, the sequence of points

$$\psi(a_1, b_1, s), \qquad \psi(a_2, b_2, s), \ldots$$

converges to

$$\psi(a, b, s),$$

as follows from the continuity of $\psi^i(u, v, s)$ in the variables u and v. Therefore no point on the path ab lies outside V, and by the preceding paragraph the point pair (a, b) is visible. Therefore the totality of visible point pairs is closed.

Further, if (a, b) is any visible pair of points, both of which are inside V, we have

$$V\{\psi(a, b, s)\} < 0,$$

for any s between 0 and 1. By the uniform continuity of $\psi^i(a, b, s)$ we have

$$V\{\psi(a+\Delta a, b+\Delta b, s)\} < 0,$$

for all small values of Δa and Δb. Therefore the set of all visible point pairs (a, b), where a and b are both in the open region C, is open. That is to say, the set of all visible point pairs in C is both open and closed relative to the $2n$-dimensional region, $C \times C$, consisting of all point pairs in C.

Since C is connected, it follows that $C \times C$ is connected. Moreover, the degenerate point pair (a, a) is visible, where a is any point in C. Therefore the set of visible point pairs is not empty, and C is convex.* Since the set of visible point pairs in the closed region $\bar{C}$ is closed, it follows that $\bar{C}$ is completely convex.

Let U be any open region in an affine space of paths, and let P be

* If a non-vacuous sub-set of a connected set X is both open and closed, relative to X, it is the set X itself. This follows at once from our definition, and is often taken as the defining property.

a given point in U. A coordinate-system exists in which a cell contained in U is represented by the region (2.2) and P by some point inside V. We have, therefore, the theorem:

If U is any open region in an affine space of paths, and if P is any point in U, there is a simple and completely convex closed region containing P and contained in U.

Another statement of this theorem is: *an affine space of paths has a set of convex regions for a fundamental set of neighbourhoods.**

In particular the theorem applies to the geodesics in a Riemannian space. *A Riemannian space with a positive ds^2 has a set of convex spheres for a fundamental set of neighbourhoods.* For any point may be taken as the origin of normal coordinates in which the locus given by

$$y^i y^i - r^2 = 0,$$

for a small enough r, is a sphere and may be taken as V.

The theorem obviously applies to projective as well as to affine spaces of paths.

* This statement refers to a topological space with an affine connexion defined at each point. For a set of axioms describing such spaces see O. Veblen and J. H. C. Whitehead, *Proc. National Academy of Sciences*, 17 (1931), 551–61, or Chap. VI of a forthcoming Cambridge Tract by the same authors.

ON DIVISOR TRANSFORMS

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1. Introduction

1.1. THE theory of Fourier transforms deals with the reciprocal relations

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu\, F(u)\, du,$$

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu f(u)\, du. \qquad (1.11)$$

If in these we put $u = (2\pi)^{\frac{1}{2}}t$, and write $\phi(x)$ for $f\{x\sqrt{(2\pi)}\}$, $\Phi(x)$ for $F\{x\sqrt{(2\pi)}\}$, we obtain the relations

$$\phi(x) = 2\int_0^\infty \cos 2\pi xt\, \Phi(t)\, dt,$$

$$\Phi(x) = 2\int_0^\infty \cos 2\pi xt\, \phi(t)\, dt. \qquad (1.12)$$

Now consider the summation formula*

$$\tfrac{1}{2}f(0) + \sum_{r=1}^\infty f(r) = \int_0^\infty f(t)\, dt + 2\sum_{m=1}^\infty \int_0^\infty \cos 2\pi mt f(t)\, dt. \qquad (1.13)$$

If, for $m = 0, 1, 2, \ldots$,

$$2\int_0^\infty \cos 2\pi mt f(t)\, dt = F(m),$$

then the right-hand side of (1.13) becomes $\frac{1}{2}F(0) + \sum F(m)$. The relations (1.12) show that, if we use the summation formula with $F(x)$ instead of $f(x)$ on the left-hand side, we get $f(x)$ instead of $F(x)$ on the right-hand side.

Next consider Voronoï's formula in the theory of divisors. We have recently given a proof † of this in which the limits of integration

* It is a form of Poisson's formula as applied to an even function; cf. Mordell, *Journal London Math. Soc.* 4 (1929), 285.

† *Quart. J. of Math.* (Oxford), 2 (1931), 31–54, § 7.

are $(0,b)$. Taking $b=\infty$, we obtain the formal equation

$$-\tfrac{1}{4}f(0)+\sum_{n=1}^{\infty}d(n)f(n)$$

$$=\int_0^\infty(2\gamma+\log t)f(t)\,dt+2\pi\sum_{n=1}^{\infty}d(n)\int_0^\infty\lambda_0\{4\pi\sqrt{(nt)}\}f(t)\,dt, \quad (1.14)$$

where $\lambda_0(x)=(2/\pi)K_0(x)-Y_0(x)$.

The reciprocal relations which we have seen to be connected with (1.13) suggest that there are reciprocal relations connected with (1.14), namely,

$$F(x)=2\pi\int_0^\infty\lambda_0\{4\pi\sqrt{(xt)}\}f(t)\,dt,$$

$$f(x)=2\pi\int_0^\infty\lambda_0\{4\pi\sqrt{(xt)}\}F(t)\,dt. \quad (1.15)$$

It is with the relations (1.15) and their generalizations that the present paper is concerned.

Hardy* has touched upon these relations and noticed (1.15) and one other special case of the results we obtain. He relates them to the divisor problem in a different manner.

1.2. Our proof of (1.15) depends upon an inversion of a repeated integral† and is a direct justification of the formal processes by which we discovered the generalized form of (1.15).

1.3. In § 5 we consider what we call a *cross-transform*, a simple example of which is that, under suitable conditions governing $f(x)$, if

$$F(x)=\int_0^\infty\left\{Y_1\big(2\sqrt{(xt)}\big)-\frac{2}{\pi}K_1\big(2\sqrt{(xt)}\big)\right\}f(t)\,dt,$$

then $$f(x)=\int_0^\infty\left\{Y_1\big(2\sqrt{(xt)}\big)+\frac{2}{\pi}K_1\big(2\sqrt{(xt)}\big)\right\}F(t)\,dt.$$

2. Preliminary formulae

2.1. The functions which occur in the following sections are modified forms of the Bessel functions

$$-Y_n(z)+\frac{2}{\pi}K_n(z), \qquad -Y_n(z)-\frac{2}{\pi}K_n(z).$$

* *Messenger of Math.* 57 (1928), 119.

† Our original method of proving that the order of integration might be changed has been replaced by a more powerful one indicated to us by Professor Titchmarsh. We gratefully acknowledge the consequent improvement in the analysis.

We use the notation of a former paper* and write

$$\lambda_\nu(z) = -Y_{/\nu}(z) - K_{/\nu}(z), \qquad \mu_\nu(z) = -Y_{/\nu}(z) + K_{/\nu}(z), \tag{2.11}$$

where

$$Y_{/\nu}(z) = \frac{1}{\pi}\sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}z)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)}\{2\log(\tfrac{1}{2}z) - \psi(m+1) - \psi(m+\nu+1)\},$$

$$K_{/\nu}(z) = \frac{1}{\pi}\sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)}\{2\log(\tfrac{1}{2}z) - \psi(m+1) - \psi(m+\nu+1)\}.$$

For convenience, we also use the notation

$$J_{/\nu}(z) = (\tfrac{1}{2}z)^{-\nu} J_\nu(z).$$

Formulae which follow directly from the expansions are†

$$\left.\begin{aligned} K_{/\nu}(z) + K_{/\nu}(-zi) &= -\lambda_\nu(z) - iJ_{/\nu}(z) \\ K_{/\nu}(z) + K_{/\nu}(zi) &= -\lambda_\nu(z) + iJ_{/\nu}(z), \end{aligned}\right\} \tag{2.12}$$

$$\left.\begin{aligned} K_{/\nu}(z) - K_{/\nu}(-zi) &= \mu_\nu(z) + iJ_{/\nu}(z) \\ K_{/\nu}(z) - K_{/\nu}(zi) &= \mu_\nu(z) - iJ_{/\nu}(z). \end{aligned}\right\} \tag{2.13}$$

In our former paper we established the asymptotic expansions, valid when $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$,

$$\lambda_\nu(z) \sim (\tfrac{1}{2}z)^{-\nu}\left\{-Y_\nu(z) + \frac{2}{\pi}e^{\nu\pi i}K_\nu(z)\right\} - \frac{2}{\pi}\sum_{r=1}^{\infty} \frac{\Gamma(2r)}{\Gamma(\nu-2r+1)}\left(\frac{2}{z}\right)^{4r}, \tag{2.14}$$

$$\mu_\nu(z) \sim (\tfrac{1}{2}z)^{-\nu}\left\{-Y_\nu(z) - \frac{2}{\pi}e^{\nu\pi i}K_\nu(z)\right\} - \frac{2}{\pi}\sum_{r=0}^{\infty} \frac{\Gamma(2r+1)}{\Gamma(\nu-2r)}\left(\frac{2}{z}\right)^{4r+2}. \tag{2.15}$$

The expansions which replace these over the range $-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi$ are obtained by writing $e^{-\nu\pi i}$ for $e^{\nu\pi i}$ in the above.

For the purposes of the present paper we wish to dispense with some of the initial terms of the asymptotic series in (2.14), (2.15), and we consider the two functions defined by the equations‡

$$\lambda_\nu^k(z) = \lambda_\nu(z) + \frac{2}{\pi}\sum_{r=1}^{k} \frac{\Gamma(2r)}{\Gamma(\nu-2r+1)}\left(\frac{2}{z}\right)^{4r}, \tag{2.16}$$

$$\mu_\nu^k(z) = \mu_\nu(z) + \frac{2}{\pi}\sum_{r=0}^{k} \frac{\Gamma(2r+1)}{\Gamma(\nu-2r)}\left(\frac{2}{z}\right)^{4r+2}. \tag{2.17}$$

* Dixon and Ferrar, *Quart. J. of Math.* (Oxford), 2 (1931), 31–54.

† We write i for $e^{\frac{1}{2}\pi i}$, $-i$ for $e^{-\frac{1}{2}\pi i}$ so that if z is real and positive,

$$\arg(-zi) = -\tfrac{1}{2}\pi.$$

‡ The series in (2.17) denotes the first $k+1$ terms of the series $\sum\limits_{r=0}^{\infty}$. In using the notation we encounter the cases $k = -1$, $k = 0$, and it is for these cases that a definite interpretation of the symbol becomes necessary.

The choice of k is determined by the particular circumstances of each problem that arises.

2.2. Let n be a positive integer or zero, p the greatest integer in $\frac{1}{2}n$, and μ, ν any real numbers. Write

$$F_\mu(z) = \lambda_\mu^p(z)+iJ_{/\mu}(z), \tag{2.21}$$

so that, as may be seen from (2.12),

$$F_\mu(zi) = \lambda_\mu^p(z)-iJ_{/\mu}(z). \tag{2.22}$$

Then,* for $z = Re^{i\theta}$ and large values of R,

$$|F_\mu(z)| = O(R^{-\mu-\frac{1}{2}})\{e^{-R\cos\theta}+e^{-R\sin\theta}\}+O(R^{-4p-4}).$$

Now consider the integral

$$\int z^{4n+1}F_\mu(az)F_\nu(bz)\,dz \qquad (a>0,\ b>0). \tag{2.23}$$

'Increasing the argument by $\frac{1}{2}\pi$' gives†

$$\int\limits_0^\infty x^{4n+1}F_\mu(ax)F_\nu(bx)\,dx = -\int\limits_0^\infty x^{4n+1}F_\mu(axi)F_\nu(bxi)\,dx, \tag{2.24}$$

provided that

(i) the integrals converge;

(ii) the integral (2.23), taken over the circular arc $z = re^{i\theta}$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, tends to zero as r tends either to zero or to infinity.

For $z = \delta e^{i\theta}$ and δ small, we have

$$|z^{2n+\frac{1}{2}}F_\mu(az)| = O(\delta^{2n+\frac{1}{2}}\log\delta)+O(\delta^{2n+\frac{1}{2}-4p}) = O(\delta^{\frac{1}{2}}\log\delta).$$

For $z = Re^{i\theta}$ and R large, we have

$$|z^{2n+\frac{1}{2}}F_\mu(az)| = O(R^{2n-\mu}e^{-aR\cos\theta})+O(R^{2n-\mu}e^{-aR\sin\theta})+O(R^{2n-4p-\frac{7}{2}}).$$

Hence the condition (ii) above is satisfied if

$$\mu,\nu > 4n-4p-\tfrac{7}{2}, \qquad \mu+\nu > 4n. \tag{2.25}$$

Moreover, for such values of μ and ν, the integrals in (2.24) converge, and hence (2.24) is valid.

When we write $F_\mu(z)$ in the form (2.21) and $F_\mu(zi)$ in the form (2.22), we see that

$$\int\limits_0^\infty x^{4n+1}\{\lambda_\mu^p(ax)+iJ_{/\mu}(ax)\}\{\lambda_\nu^p(bx)+iJ_{/\nu}(bx)\}\,dx$$

* $F_\mu(z)$ may, for purposes of getting its asymptotic expansion, be considered as a linear combination of

$$(\tfrac{1}{2}z)^{-\mu}H_\mu^{(1)}(z), \qquad (\tfrac{1}{2}z)^{-\mu}K_\mu(z),$$

and the series of (2.14) assumed to begin at $r = p+1$.

† Dixon and Ferrar, *Quart. J. of Math.* (Oxford), 1 (1930), 122–45.

changes sign when we write $-i$ for i in the integrand. Accordingly,

$$\int_0^\infty x^{4n+1}\{\lambda_\mu^p(ax)\lambda_\nu^p(bx)-J_{|\mu}(ax)J_{|\nu}(bx)\}\,dx = 0,$$

and, provided that the separate integrals converge,*

$$\int_0^\infty x^{4n+1}\lambda_\mu^p(ax)\lambda_\nu^p(bx)\,dx = \int_0^\infty x^{4n+1}J_{|\mu}(ax)J_{|\nu}(bx)\,dx. \tag{2.26}$$

This last integral is the Weber-Schafheitlin discontinuous integral, whose values are known in terms of hypergeometric functions.

2.3. It is readily proved that, if $\alpha > -1$,

$$\frac{d}{dt}\{t^{\alpha+1}\lambda_{\alpha+1}(a\sqrt{t})\} = t^\alpha\lambda_\alpha(a\sqrt{t}), \tag{2.31}$$

and that, if $\alpha > 2k-1$,

$$\frac{d}{dt}\{t^{\alpha+1}\lambda_{\alpha+1}^k(a\sqrt{t})\} = t^\alpha\lambda_\alpha^k(a\sqrt{t}). \tag{2.32}$$

By considering that

$$x^{2n+\frac{1}{2}}\lambda_{2n}^p(ax), \qquad x^{2n+\frac{3}{2}}\lambda_{2n+1}^p(ax)$$

remain finite as $x \to \infty$ and tend to zero as $x \to 0$, we see that there is a finite $A(n)$ such that, for $x > 0$,

$$|x^{2n+\frac{1}{2}}\lambda_{2n}^p(ax)|, \qquad |x^{2n+\frac{3}{2}}\lambda_{2n+1}^p(ax)| < A. \tag{2.33}$$

Moreover, for small values of x we have, by considering the series given in 2.1,

$$\lambda_{2n}^p(x) = O(|\log x|)+O(x^{-2n}),$$
$$\lambda_{2n+1}^p(x) = O(|\log x|)+O(x^{-2n}).$$

In order to deal with the two cases (i) $n > 0$, (ii) $n = 0$ simultaneously, we use the forms

$$\lambda_{2n}^p(x) = O(x^{-2n-2\delta}), \qquad \lambda_{2n+1}^p(x) = O(x^{-2n-2\delta}), \tag{2.34}$$

wherein δ is positive and as small as we please. This is the most convenient form for the detailed analysis. When formulating conditions which govern the final result, we make use of the forms,

$$\lambda_{2n}^p(x), \qquad \lambda_{2n+1}^p(x) = O(x^{-2n}|\log x|), \tag{2.35}$$

it being understood that the $|\log x|$ term may be omitted if $n > 0$.

* In our paper, *Quart. J. of Math.* (Oxford), 2 (1931), this point was not brought out.

3. The first divisor transform.

3.1. The relation which we set out to prove, subject to appropriate conditions on $f(x)$, is that if $x > 0$ and

$$F(x) = (2\pi)^{2n+1} \int_0^\infty t^{2n} \lambda_{2n}^p \{4\pi\sqrt{(xt)}\} f(t)\, dt, \tag{3.11}$$

then
$$f(x) = (2\pi)^{2n+1} \int_0^\infty t^{2n} \lambda_{2n}^p \{4\pi\sqrt{(xt)}\} F(t)\, dt. \tag{3.12}$$

This is the form which is suggested by (1.15) and its direct relation with Voronoï's summation-formula.

We shall, however, first make a change of variables, similar to that given for Fourier transforms, and prove that if

$$F(x) = \int_0^\infty t^{2n} \lambda_{2n}^p \{2\sqrt{(xt)}\} f(t)\, dt, \tag{3.13}$$

then
$$f(x) = \int_0^\infty t^{2n} \lambda_{2n}^p \{2\sqrt{(xt)}\} F(t)\, dt. \tag{3.14}$$

One obvious method of trying to build up a proof is to consider

$$\int_0^\infty t^{2n} \lambda_{2n}^p \{2\sqrt{(xt)}\}\, dt \int_0^\infty \theta^{2n} \lambda_{2n}^p \{2\sqrt{(t\theta)}\} f(\theta)\, d\theta, \tag{3.15}$$

and to show, by inverting the order of integration, that it equals $f(x)$. This would lead to the non-convergent integral

$$\int_0^\infty J_{2n}(2t\sqrt{x}) J_{2n}(2t\sqrt{\theta}) t\, dt,$$

a difficulty which has its analogue in the theory of the Fourier repeated integral. Accordingly, we first integrate by parts in the inner integral of (3.15) and note that, in virtue of (2.32),

$$\int_0^\infty \theta^{2n} \lambda_{2n}^p \{2\sqrt{(t\theta)}\} f(\theta)\, d\theta = -\int_0^\infty \theta^{2n+1} \lambda_{2n+1}^p \{2\sqrt{(t\theta)}\} f'(\theta)\, d\theta \tag{3.16}$$

whenever, for a fixed t,

$$\lim_{\theta \to 0} \theta^{2n+1} \lambda_{2n+1}^p \{2\sqrt{(t\theta)}\} f(\theta) = 0 = \lim_{\theta \to \infty} \theta^{2n+1} \lambda_{2n+1}^p \{2\sqrt{(t\theta)}\} f(\theta). \tag{3.17}$$

We shall consider, then,

$$-\int_0^\infty t^{2n} \lambda_{2n}^p \{2\sqrt{(tx)}\}\, dt \int_0^\infty \theta^{2n+1} \lambda_{2n+1}^p \{2\sqrt{(t\theta)}\} f'(\theta)\, d\theta. \tag{3.18}$$

If we write t^2 instead of t and assume that the order of integration can be changed, this becomes

$$-2\int_0^\infty \theta^{2n+1}f'(\theta)\,d\theta \int_0^\infty t^{4n+1}\lambda^p_{2n}(2t\sqrt{x})\lambda^p_{2n+1}(2t\sqrt{\theta})\,dt. \tag{3.19}$$

But the t-integral is, by (2.26),

$$\int_0^\infty \frac{J_{2n}(2t\sqrt{x})J_{2n+1}(2t\sqrt{\theta})}{x^n\theta^{n+\frac{1}{2}}}\,dt$$

and the values of this integral are known to be

$$0, \qquad \tfrac{1}{4}\theta^{-2n-1}, \qquad \tfrac{1}{2}\theta^{-2n-1},$$

according as $x >, =, < \theta$.

This gives, as the value of (3.18),

$$-\int_x^\infty f'(\theta)\,d\theta,$$

which, with suitable hypotheses, is $f(x)$.

Hence, provided we can justify the various steps of the foregoing process by placing suitable conditions on $f(x)$, the repeated integral (3.15) has the value $f(x)$.

We proceed to consider the steps in greater detail.

3.2. *The change of order of integration.*

We shall assume that $f(x)$ is an indefinite integral whose differential coefficient $f'(x)$ is such that

$$\int_0^\infty \theta^{n+\frac{1}{2}}|f'(\theta)|\,d\theta \tag{3.21}$$

is convergent.

Our immediate object is to show that, in the repeated integral

$$\int_0^\infty \theta^{2n+1}f'(\theta)\,d\theta \int_0^\infty t^{4n+1}\lambda^p_{2n}(2t\sqrt{x})\lambda^p_{2n+1}(2t\sqrt{\theta})\,dt, \tag{3.22}$$

the order of integration can be inverted.

3.21. We first show that the inversion is possible when the limits of integration are $(0,\theta_1)$ for θ and $(0,T)$ for t.

For convenience in setting out the work, we write

$$U \equiv t^{4n+1}\theta^{2n+1}\lambda^p_{2n}(2t\sqrt{x})\lambda^p_{2n+1}(2t\sqrt{\theta})f'(\theta).$$

Let η be a small positive number. Then, *irrespective of the order of integration*, we have, on applying the inequalities (2.33) to the function of $t\sqrt{\theta}$ and the inequalities (2.34) to the function of $t\sqrt{x}$,

$$\left|\int\limits_{t=0}^{\eta}\int\limits_{\theta=0}^{\theta_1} U\,d\theta dt\right| < A\int\limits_{t=0}^{\eta}\int\limits_{\theta=0}^{\theta_1} t^{-\frac{1}{2}-\delta}\theta^{n+\frac{1}{4}}|f'(\theta)|\,d\theta dt$$
$$= O(\eta^{\frac{1}{2}-\delta}).$$

Hence it is sufficient to prove inversion when the limits of integration are $(0,\theta_1)$ for θ and (η, T) for t. We have, on using (2.33),

$$\left|\int\limits_{\eta}^{T} U\,dt\right| < A\theta^{n+\frac{1}{4}}|f'(\theta)|\int\limits_{\eta}^{T} t^{-1}\,dt,$$

and hence, by the hypothesis that the integral (3.21) is convergent, the inversion is justified.*

3.22. Now suppose (we shall prove it later) that $\phi(\theta, T)$, where

$$\phi(\theta, T) = \int\limits_0^T \theta^{n+\frac{3}{4}}t^{4n+1}\lambda_{2n}^{p}(2t\sqrt{x})\lambda_{2n+1}^{p}(2t\sqrt{\theta})\,dt,$$

is bounded for all positive values of θ and T. Then, by Lebesgue's convergence theorem on the integration of sequences,†

$$\lim_{T\to\infty}\int\limits_0^\infty \theta^{n+\frac{1}{4}}f'(\theta)\phi(\theta, T)\,d\theta = \int\limits_0^\infty \theta^{n+\frac{1}{4}}f'(\theta)\phi(\theta,\infty)\,d\theta,$$

since (3.21) converges.

Hence, U being defined as in 3.21,

$$\lim_{T\to\infty}\lim_{\theta_1\to\infty}\int\limits_0^{\theta_1}d\theta\int\limits_0^T U\,dt = \int\limits_0^\infty d\theta\int\limits_0^\infty U\,dt.$$

But, on using § 3.21 to invert the order of integration in the left-hand side, this gives

$$\int\limits_0^\infty dt\int\limits_0^\infty U\,d\theta = \int\limits_0^\infty d\theta\int\limits_0^\infty U\,dt.$$

Hence, we may invert the order of integration in (3.22) if we can prove that $\phi(\theta, T)$ is bounded.‡

* Cf. Hobson, *Functions of a Real Variable*, vol. ii, p. 339.

† Cf. ibid., vol. ii, § 202.

‡ Cf. Hardy and Titchmarsh, *Proc. London Math. Soc.* (2) 23 (1925), p. 10; they refer to their method as 'inversion by bounded convergence'.

3.3. *Proof that $\phi(\theta, T)$ is bounded.*

In addition to the inequalities (2.33) and (2.34) we shall need, for this proof, the leading terms in the asymptotic expansion of the product of Bessel functions which occurs in the definition

$$\phi(\theta, T) = \int_0^T \theta^{n+\frac{3}{4}} t^{4n+1} \lambda_{2n}^p(2t\sqrt{x}) \lambda_{2n+1}^p(2t\sqrt{\theta})\, dt. \tag{3.31}$$

When n is odd,* we have, for large values of t and $t\sqrt{\theta}$,

$$\lambda_{2n}^p(2t\sqrt{x}) = At^{-2n-\frac{1}{2}} \sin\{2t\sqrt{x}-(n+\tfrac{1}{4})\pi\} + O(t^{-2n-\frac{3}{2}}),$$

$$\lambda_{2n+1}^p(2t\sqrt{\theta}) = B(t\sqrt{\theta})^{-2n-\frac{3}{2}} \sin\{2t\sqrt{\theta}-(n+\tfrac{3}{4})\pi\} + O(\theta^{-n-1}t^{-2n-2}),$$

so that the product of the two functions is of the form

$$t^{-4n-2}\theta^{-n-\frac{3}{4}}[C\sin\{2t(\sqrt{x}-\sqrt{\theta})\}+D\cos\{2t(\sqrt{x}+\sqrt{\theta})\}]+$$
$$+O(\theta^{-n-1}t^{-4n-\frac{5}{2}})+O(\theta^{-n-\frac{3}{4}}t^{-4n-3}). \tag{3.32}$$

We proceed to consider separately the three ranges of values (i) θ small, (ii) $A_1 < \theta < A_2$ where A_1, A_2 are any finite positive numbers, (iii) θ large.

3.31. *Small values of θ*; say, $0 < \theta \leqslant A_1$.

We write†

$$\phi(\theta, T) = \int_0^1 + \int_1^{1/\sqrt{\theta}} + \int_{1/\sqrt{\theta}}^T = \alpha_1+\alpha_2+\alpha_3,$$

say, and assume for the moment that $T \geqslant \theta^{-\frac{1}{2}}$. In α_1, $0 \leqslant t \leqslant 1$, and if δ is an arbitrarily small positive number,

$$\lambda_{2n}^p(2t\sqrt{x}) = O(t^{-2n-2\delta}), \qquad \lambda_{2n+1}^p(2t\sqrt{\theta}) = O(t^{-2n-2\delta}\theta^{-n-\delta}),$$

$$|\alpha_1| = O\int_0^1 \theta^{\frac{3}{4}-\delta}t^{1-4\delta}\, dt = O(1). \tag{i}$$

In α_2, $0 \leqslant t\sqrt{\theta} \leqslant 1$, $t \geqslant 1$,

$$\lambda_{2n}^p(2t\sqrt{x}) = O(t^{-2n-\frac{1}{2}}), \qquad \lambda_{2n+1}^p(2t\sqrt{\theta}) = O(t^{-2n-2\delta}\theta^{-n-\delta}),$$

$$|\alpha_2| = O\int_1^{1/\sqrt{\theta}} \theta^{\frac{3}{4}-\delta}t^{\frac{1}{2}-2\delta}\, dt = O(1). \tag{ii}$$

* The case n odd is the less favourable; when n is even the second of the order terms is $O(\theta^{-n-\frac{5}{4}}t^{-2n-\frac{5}{2}})$. We use A, B, C, D to denote constants independent of θ and t.

† The integrand of (3.31) is to be understood in all the 'shadow' integrals of this and the following sub-sections.

In α_3, $t \geqslant \theta^{-\frac{1}{2}}$, $t\sqrt{\theta} \geqslant 1$, and, on using the asymptotic form (3.32), we have

$$\alpha_3 = C \int\limits_{1/\sqrt{\theta}}^{T} \sin\{2t(\sqrt{x}-\sqrt{\theta})\}\frac{dt}{t}+D\int\limits_{1/\sqrt{\theta}}^{T} \cos\{2t(\sqrt{x}+\sqrt{\theta})\}\frac{dt}{t}+$$
$$+O\int\limits_{1/\sqrt{\theta}}^{T} \theta^{-\frac{1}{2}}t^{-\frac{3}{2}}\,dt+O\int\limits_{1/\sqrt{\theta}}^{T} t^{-2}\,dt.$$

On integrating the first two of these integrals by parts, we see that

$$|\alpha_3| = O(\theta^{\frac{1}{2}})+O(1) = O(1). \tag{iii}$$

When $T < \theta^{-\frac{1}{2}}$, (iii) is not necessary, while modified forms of (i) and (ii) prove the boundedness of $\phi(\theta, T)$. Hence, for $T > 0$, the results (i), (ii), (iii) of this sub-section prove that

'$\phi(\theta, T)$ is bounded when $0 < \theta \leqslant A_1$, and A_1 is small'.

3.32. *Moderate values of* θ; say, $A_1 \leqslant \theta \leqslant A_2$.

We write
$$\phi(\theta, T) = \int\limits_0^1 + \int\limits_1^T = \beta_1+\beta_2,$$
say, and assume for the moment that $T \geqslant 1$.

In β_1 we proceed as in α_1 and obtain

$$|\beta_1| = O\int\limits_0^1 t^{1-4\delta}\,dt = O(1). \tag{i}$$

In β_2 we use the asymptotic expansions and, as in α_3, obtain an expression for β_2 as the sum of four integrals. In the first integral put $2t|\sqrt{x}-\sqrt{\theta}| = y$, and in the second put $2t(\sqrt{x}+\sqrt{\theta}) = y$. It is then easily seen that β_2 is bounded.

Further, when $T < 1$, $\phi(\theta, T)$ is bounded in virtue of (i) alone. Hence, for $T > 0$, we have proved that

'$\phi(\theta, T)$ is bounded when $A_1 \leqslant \theta \leqslant A_2$'.

3.33. *Large values of* θ; say $\theta > A_2$.

We write
$$\phi(\theta, T) = \int\limits_0^{1/\sqrt{\theta}} + \int\limits_{1/\sqrt{\theta}}^1 + \int\limits_1^T = \gamma_1+\gamma_2+\gamma_3,$$
say, and assume for the moment that $T \geqslant 1$.

In γ_1, $t \leqslant \theta^{-\frac{1}{2}}$, $t\sqrt{\theta} \leqslant 1$,

$$\lambda^p_{2n}(2t\sqrt{x}) = O(t^{-2n-2\delta}), \qquad \lambda^p_{2n+1}(2t\sqrt{\theta}) = O(t^{-2n-2\delta}\theta^{-n-\delta}),$$

$$|\gamma_1| = O\int\limits_0^{1/\sqrt{\theta}} \theta^{\frac{1}{2}-\delta}t^{1-4\delta}\,dt = O(\theta^{-\frac{1}{2}+\delta}). \tag{i}$$

In γ_2, $t \leqslant 1$, $t\surd\theta \geqslant 1$,

$$\lambda_{2n}^{p}(2t\surd x) = O(t^{-2n-2\delta}), \qquad \lambda_{2n+1}^{p}(2t\surd\theta) = O(t^{-2n-\frac{3}{2}}\theta^{-n-\frac{3}{4}}),$$

$$|\gamma_2| = O\int_{1/\surd\theta}^{1} t^{-\frac{1}{2}-2\delta}\,dt = O(\theta^{-\frac{1}{4}+\delta}). \tag{ii}$$

In γ_3, $t \geqslant 1$, $t\surd\theta \geqslant \surd A_2$, and we may use the asymptotic expansion (3.32). This procedure, as in the case of α_3 and β_2, leads to the result

$$|\gamma_3| = O(1). \tag{iii}$$

If $T < 1$, (iii) is not necessary and (i) and (ii), suitably modified, prove the boundedness of $\phi(\theta, T)$.

Hence we have proved, by the aid of (i), (ii), (iii) of this subsection, that

'$\phi(\theta, T)$ is bounded when $\theta > A_2$'.

3.4. We have thus proved that, in the repeated integral (3.22), the order of integration can be inverted. Returning now to § 3.1, we see that, under the two conditions

[A] $f(\theta)$ is an indefinite integral whose differential coefficient $f'(\theta)$ is such that

$$\int_0^\infty \theta^{n+\frac{1}{2}}|f'(\theta)|\,d\theta$$

converges;

[B] $\qquad f(\theta) \to 0, \qquad \text{as } \theta \to \infty,$

we have proved that

$$f(x) = -\int_x^\infty f'(\theta)\,d\theta$$

$$= -\int_0^\infty t^{2n}\lambda_{2n}^{p}\{2\surd(tx)\}\,dt \int_0^\infty \theta^{2n+1}\lambda_{2n+1}^{p}\{2\surd(t\theta)\}f'(\theta)\,d\theta.$$

If, further, we impose the conditions*

[B₁] $\qquad \theta^{n+\frac{1}{2}}f(\theta) \to 0, \qquad \text{as } \theta \to \infty;$

$\qquad \theta^{n+1}|\log\theta|f(\theta) \to 0, \qquad \text{as } \theta \to 0,$

then the conditions (3.17) are satisfied and we have

$$f(x) = \int_0^\infty t^{2n}\lambda_{2n}^{p}\{2\surd(tx)\}\,dt \int_0^\infty \theta^{2n}\lambda_{2n}^{p}\{2\surd(t\theta)\}f(\theta)\,d\theta.$$

Accordingly, we have proved that (3.14) is a consequence of (3.13) whenever the conditions [A] and [B₁] are satisfied.

* The term $|\log\theta|$ may be dropped if $n > 0$; we have used (2.35).

4. A second divisor transform

4.1. *The integral on which this transform is based.*

If n is odd, let $q = \frac{1}{2}(n-1)$, and if n is even, let $q = \frac{1}{2}n-1$. Write now

$$\phi_\nu(z) = \mu_\nu^q(z)+iJ_{/\nu}(z). \tag{4.11}$$

where the function $\mu_\nu^q(z)$ has been defined in (2.17); then, by (2.13),

$$\phi_\nu(zi) = -\mu_\nu^q(z)+iJ_{/\nu}(z). \tag{4.12}$$

We may work through the argument of 2.2 with $\phi_\nu(z)$ instead of $F_\mu(z)$ and obtain the result

$$\int\limits_0^\infty x^{4n+1}\mu_\mu^q(ax)\mu_\nu^q(bx)\,dx = \int\limits_0^\infty x^{4n+1}J_{/\mu}(ax)J_{/\nu}(bx)\,dx \tag{4.13}$$

provided that
$$\mu,\nu > 4n-4q-\tfrac{11}{2}, \qquad \mu+\nu > 4n,$$
and the integrals converge.

4.2. Just as in 2.3 we have a set of formulae

$$\frac{d}{dt}\{t^{\alpha+1}\mu_{\alpha+1}^k(a\sqrt{t})\} = t^\alpha\mu_\alpha^k(a\sqrt{t}) \qquad (\alpha > 2k), \tag{4.21}$$

$$|x^{2n+\frac{1}{2}}\mu_{2n}^q(ax)|, \qquad |x^{2n+\frac{3}{2}}\mu_{2n+1}^q(ax)| < A. \tag{4.22}$$

The choice of q instead of p is, in fact, dictated by the desirability of having (4.22) similar to (2.33). For $x = \delta$ we have, when n is even (not zero),
$$\delta^{2n+\frac{1}{2}}\mu_{2n}^p(\delta) = O(\delta^{-\frac{3}{2}}),$$
so that p instead of q would render false the inequality (4.22).

Further, for small values of x, we have
$$\mu_{2n}^q(x) = O(x^{-2n}), \qquad O(x^{-2n+2}), \qquad O(x^2|\log x|),$$
according as n is odd, even, or zero. We see then that for all values of n we may write

$$\mu_{2n}^q(x) = O(x^{-2n}), \qquad \mu_{2n+1}^q(x) = O(x^{-2n}). \tag{4.23}$$

It should be noticed that these order-terms are those of (2.34) with $\delta = 0$.

4.3. If now we work through § 3 with μ_{2n}^q instead of λ_{2n}^p, we see that the sole difference in the details is that we may put $\delta = 0$, and this does not affect the argument.

Accordingly, if the conditions [A] and [B] of 3.4 are satisfied, we have

$$f(x) = -\int\limits_0^\infty t^{2n}\mu_{2n}^q\{2\sqrt{(tx)}\}\,dt \int\limits^\infty \theta^{2n+1}\mu_{2n+1}^q\{2\sqrt{(t\theta)}\}f'(\theta)\,d\theta. \tag{4.31}$$

If we impose the further conditions

$$[\mathrm{B}_2] \qquad \theta^{n+\frac{1}{2}}f(\theta) \to 0, \qquad \text{as } \theta \to \infty,$$
$$\theta^{n+1}f(\theta) \to 0, \qquad \text{as } \theta \to 0,$$

then, for any fixed t, we have

$$\theta^{2n+1}\mu^q_{2n+1}\{2\sqrt{(\theta t)}\}f(\theta) \to 0 \tag{4.32}$$

both as $\theta \to 0$ and as $\theta \to \infty$.

Accordingly, if the conditions [A] of 3.4 and $[\mathrm{B}_2]$ are satisfied, then

$$f(x) = \int\limits_0^\infty t^{2n}\mu^q_{2n}\{2\sqrt{(tx)}\}\,dt \int\limits_0^\infty \theta^{2n}\mu^q_{2n}\{2\sqrt{(t\theta)}\}f(\theta)\,d\theta, \tag{4.33}$$

i.e., under the above conditions, if

$$F(x) = \int\limits_0^\infty t^{2n}\mu^q_{2n}\{2\sqrt{(xt)}\}f(t)\,dt,$$

then

$$f(x) = \int\limits_0^\infty t^{2n}\mu^q_{2n}\{2\sqrt{(xt)}\}F(t)\,dt. \tag{4.34}$$

5. A cross-transform

5.1. We now show that, subject to suitable conditions governing $f(x)$, the functions λ and μ have the following property. If

$$F(x) = \int\limits_0^\infty t^{2n+1}\mu^p_{2n+1}\{2\sqrt{(xt)}\}f(t)\,dt, \tag{5.11}$$

then

$$f(x) = \int\limits_0^\infty t^{2n+1}\lambda^p_{2n+1}\{2\sqrt{(xt)}\}F(t)\,dt. \tag{5.12}$$

The proof of this relation between $f(t)$ and $F(t)$ depends, as did the previous ones, on finding a double integral which reduces to $\int f'(\theta)\,d\theta$.

As we have already seen, if

$$\left.\begin{aligned} F_\mu(ax) &= \lambda^p_\mu(ax)+iJ_{|\mu}(ax), \\ \phi_\nu(bx) &= \mu^p_\nu(bx)+iJ_{|\nu}(bx), \end{aligned}\right\} \tag{5.13}$$

then

$$\left.\begin{aligned} F_\mu(axi) &= \lambda^p_\mu(ax)-iJ_{|\mu}(ax), \\ \phi_\nu(bxi) &= -\mu^p_\nu(bx)+iJ_{|\nu}(bx). \end{aligned}\right\} \tag{5.14}$$

Consider the integral

$$\int z^{4n+3}F_\mu(az)\phi_\nu(bz)\,dz \qquad (a > 0,\ b > 0),$$

We find the process of 'increasing the argument by $\frac{1}{2}\pi$' to be valid if

either, (i) n is even, $\nu > 2n-\frac{3}{2}$, $\mu > 2n-\frac{7}{2}$, $\mu+\nu > 4n+2$,

or, (ii) n is odd, $\nu > 2n+\frac{1}{2}$, $\mu > 2n-\frac{3}{2}$, $\mu+\nu > 4n+2$.

This process gives, in virtue of (5.13) and (5.14),

$$\int_0^\infty \{\lambda_\mu^p(ax)+iJ_{/\mu}(ax)\}\{\mu_\nu^p(bx)+iJ_{/\nu}(bx)\}x^{4n+3}\,dx$$

$$= \int_0^\infty \{\lambda_\mu^p(ax)-iJ_{/\mu}(ax)\}\{-\mu_\nu^p(bx)+iJ_{/\nu}(bx)\}x^{4n+3}\,dx,$$

and so

$$\int_0^\infty \lambda_\mu^p(ax)\mu_\nu^p(bx)x^{4n+3}\,dx = \int_0^\infty J_{/\mu}(ax)J_{/\nu}(bx)x^{4n+3}\,dx, \tag{5.15}$$

provided the last integral is convergent and μ, ν satisfy the conditions given above.

5.2. Consider now the repeated integral

$$\int_0^\infty t^{2n+1}\lambda_{2n+1}^p\{2\surd(tx)\}\,dt \int_0^\infty \theta^{2n+2}\mu_{2n+2}^p\{2\surd(t\theta)\}f'(\theta)\,d\theta. \tag{5.21}$$

Write t^2 for t, and assume that the order of integration can be changed; we get

$$2\int_0^\infty \theta^{2n+2}f'(\theta)\,d\theta \int_0^\infty t^{4n+3}\lambda_{2n+1}^p(2t\sqrt{x})\mu_{2n+2}^p(2t\sqrt{\theta})\,dt.$$

The t-integral is, by (5.15),

$$\int_0^\infty \frac{J_{2n+1}(2t\sqrt{x})J_{2n+2}(2t\sqrt{\theta})}{x^{n+\frac{1}{2}}\theta^{n+1}}\,dt,$$

whose values are 0, $\frac{1}{4}\theta^{-2n-2}$, $\frac{1}{2}\theta^{-2n-2}$

according as $x >, =, < \theta$.

Hence (5.21) is equal to

$$\int_x^\infty f'(\theta)\,d\theta.$$

5.3. The condition necessary to justify the inversion of the repeated integral (5.21) is that

$$\int_0^\infty \theta^{n+\frac{3}{4}}|f'(\theta)|\,d\theta$$

should be convergent.

The details necessary to the proof are very similar to those in §§ 3.2, 3.3. The inequalities (2.33) and (2.34) deal with the λ-function, while, for the μ-function, we have

$$|x^{2n+\frac{3}{2}}\mu^p_{2n+2}(x)| < A, \text{ for all } x,$$
$$\mu^p_{2n+2}(x) = O(x^{-2n-2}), \text{ for small } x.$$

We omit the details of the proof.

5.4. We have then, if $f(x)$ is an indefinite integral and tends to zero as x tends to infinity,

$$f(x) = -\int_0^\infty t^{2n+1}\lambda^p_{2n+1}\{2\sqrt{(tx)}\}\,dt\int_0^\infty \theta^{2n+2}\mu^p_{2n+2}\{2\sqrt{(t\theta)}\}f'(\theta)\,d\theta.$$

Integrate by parts in the θ-integral and use (4.21). We see that, if we impose the further conditions

$$\theta^{n+1}f(\theta) \to 0, \qquad \text{as } \theta \to 0,$$
$$\theta^{n+\frac{3}{4}}f(\theta) \to 0, \qquad \text{as } \theta \to \infty,$$

we may write

$$f(x) = \int_0^\infty t^{2n+1}\lambda^p_{2n+1}\{2\sqrt{(tx)}\}\,dt\int_0^\infty \theta^{2n+1}\mu^p_{2n+1}\{2\sqrt{(t\theta)}\}f(\theta)\,d\theta.$$

Hence (5.12) is a consequence of (5.11) whenever both

[A′] $f(\theta)$ is an indefinite integral whose differential coefficient $f'(\theta)$ is such that

$$\int_0^\infty \theta^{n+\frac{3}{4}}|f'(\theta)|\,d\theta$$

converges;

[B′] $$\theta^{n+\frac{3}{4}}f(\theta) \to 0, \qquad \text{as } \theta \to \infty,$$
$$\theta^{n+1}f(\theta) \to 0, \qquad \text{as } \theta \to 0\,.$$

5.5. We can, by placing further restrictions on $f(x)$, show that λ and μ may be interchanged in the equations (5.11) and (5.12). That such an interchange is possible is obvious from the equations themselves, though these give little indication as to what conditions to impose on $f(x)$ when the interchange is made.

By using (5.15) we easily obtain

$$\int_x^\infty f'(\theta)\,d\theta = \int_0^\infty \theta^{2n+2}f'(\theta)\,d\theta\int_0^\infty t^{2n+1}\mu^p_{2n+1}\{2\sqrt{(tx)}\}\lambda^p_{2n+2}\{2\sqrt{(t\theta)}\}\,dt.$$

When we come to the inversion of the order of integration we at once encounter the difficulty that, for small x, and positive even values of n,

$$x^{2n+\frac{3}{2}}\mu^p_{2n+1}(x) = O(x^{-\frac{1}{2}}).$$

The analogue of (2.33) and (4.22) is not immediately available. The exact determination of the conditions to impose on $f(x)$ in order to counteract this irregularity would, if the need arose, be easy. We shall not here enter into their detailed discussion.

6. Change of Notation

We have given our results in terms of the λ, μ functions because we have had occasion, in a previous paper, to collect, in a form which used this particular notation, most of the facts we require for our present purpose. The functions

$$\lambda_n^k(z), \qquad \mu_n^k(z)$$

can, however, when n is an integer, be regarded as modified forms of the functions

$$-Y_n(z) \pm \frac{2}{\pi} K_n(z).$$

As an example, take the functions which occur in the first divisor transform. Here*

$$(\tfrac{1}{2}z)^{2n}\lambda_{2n}(z) = -Y_{2n}(z) + \frac{2}{\pi} K_{2n}(z) - \frac{2}{\pi}\sum_{r=1}^{n} \frac{\Gamma(2r)}{\Gamma(2n-2r+1)} (\tfrac{1}{2}z)^{2n-4r},$$

so that

$$(\tfrac{1}{2}z)^{2n}\lambda_{2n}^p(z) = -Y_{2n}(z) + \frac{2}{\pi} K_{2n}(z) - \frac{2}{\pi}\sum_{r=p+1}^{n} \frac{\Gamma(2r)}{\Gamma(2n-2r+1)} (\tfrac{1}{2}z)^{2n-4r},$$

or, on writing $n-r = m$,

$$-Y_{2n}(z) + \frac{2}{\pi} K_{2n}(z) - \frac{2}{\pi}\sum_{m=0}^{n-1-p} \frac{\Gamma(2n-2m)}{\Gamma(2m+1)} (\tfrac{1}{2}z)^{-2n+4m}.$$

If we compare this with the expansions of Y_n and K_n as they are given in Watson, *Theory of Bessel Functions*, we see that

$$(\tfrac{1}{2}z)^{2n}\lambda_{2n}^p(z)$$

is the function
$$-Y_{2n}(z) + \frac{2}{\pi} K_{2n}(z)$$

deprived of those terms in its expansion which involve negative powers of z.

In conclusion, it is perhaps worth while to call attention to the following points.

There is a certain degree of indetermination in the functions which

* Dixon and Ferrar, *Quart. J. of Math.* (Oxford), 2 (1931), 41–2, § 3.5.

may occur in the preceding transforms. For example, instead of using the functions $\lambda_{2n}^{p}(x)$, $\mu_{2n}^{q}(x)$ we may, within certain ranges of values of k use functions $\lambda_{2n}^{k}(x)$, $\mu_{2n}^{k}(x)$ where k is not necessarily p or q as they are defined in §§ 2.2 and 4.1. We have chosen, in each case, a value of k which made it possible to state one simple set of conditions which controlled the analysis both when n was odd and when n was even.

Finally, if we consider the double-summation formula

$$\sum_{r,s=-\infty}^{\infty} f(r,s) = \sum_{m,n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(mu+nv)} f(u,v)\, du dv$$

for the particular case when $f(r,s)$ is a function of the product rs only, we see that our transforms are, in fact, special cases of double Fourier transforms. The formulae required to reduce the double Fourier integral to a single integral involving Bessel functions are given by Watson in his *Theory of Bessel Functions*.*

* p. 184, § 6.23.

NOTES ON THERMODYNAMICS

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IV. THE THEORY OF EQUILIBRIUM CONSTANTS AND A GENERAL THERMODYNAMIC FORMULA

1. THE theory of the equilibrium box, the derivation of the van 't Hoff isochore, its integration and the connexion of the constant of integration with vapour-pressure constants by means of Nernst's Heat Theorem are now classical. The applications of the theory in practice, especially to systems not necessarily composed of dilute phases, are to be found, for example, in the treatises of Lewis and Randall and of Hinshelwood. The following treatment is based entirely on the ideas set out in Lewis and Randall's *Thermodynamics*. But the mathematician likes to put things in his own language, and the first aim of the present note was to express the results in a form which might appeal to mathematicians interested in the subject. But a general formula then emerges which has aspects of novelty, and which contains a great many known results, usually deduced separately, as particular cases. The power of the formula arises from its flexibility, a flexibility which only comes into evidence when the more standardized definitions of equilibrium constants, etc., used by physical chemists are replaced by expressions having a greater freedom of choice of certain constants. This seems to justify a rigorous mathematical treatment of what might be considered otherwise an exhausted subject.

The notation throughout closely follows that of Lewis and Randall, but it is adapted in parts to that of Willard Gibbs.

2. We enumerate first certain well-known thermodynamic properties of homogeneous substances which we require later. Let U_r be the energy, S_r the entropy, v_r the volume per gram-molecule, at temperature T and pressure p, of a particular homogeneous substance which we identify by the suffix r. Let ρ_r $(= 1/v_r)$ be its density in gram-molecules per cm³. Let C_r be its chemical formula. The heat-function H_r and the free-energy function F_r are defined by the relations

$$H_r = U_r + pv_r \tag{1}$$

$$F_r = H_r - S_r T. \tag{2}$$

Adopting p, T as the independent canonically conjugate variables we have

$$dF = -S_r dT + v_r dp,$$

whence
$$\frac{\partial F_r}{\partial T} = -S_r, \qquad \frac{\partial F_r}{\partial p} = v_r, \qquad -\frac{\partial S_r}{\partial p} = \frac{\partial v_r}{\partial T}. \tag{3}$$

From these we have

$$\frac{\partial H_r}{\partial p} = \frac{\partial (F_r + TS_r)}{\partial p} = v_r - T\frac{\partial v_r}{\partial T} = -T^2 \frac{\partial}{\partial T}\left(\frac{v_r}{T}\right) \tag{3'}$$

and
$$\frac{\partial F_r}{\partial T} = -\frac{H_r - F_r}{T},$$

or
$$\frac{\partial}{\partial T}\left(\frac{F_r}{T}\right) = -\frac{H_r}{T^2}. \tag{3''}$$

Further, if c_p^r denotes the gram-molecular specific heat at constant pressure,

$$c_p^r = \frac{\partial H_r}{\partial T} = \frac{\partial (F_r + TS_r)}{\partial T} = -T\frac{\partial^2 F_r}{\partial T^2}, \tag{3'''}$$

$$\frac{\partial c_p^r}{\partial p} = -T\frac{\partial^2 v_r}{\partial T^2}. \tag{3^{iv}}$$

3. Let the equation of a chemical reaction be written in the form

$$\sum \nu_r C_r = 0,$$

where the ν_r's are integers, positive for products and negative for reactants. Consider a chemical change in which the correctly-proportioned stoicheometric quantities of the reactants in given phases are completely converted into the products. Let the reactants, before the reaction, be in the form of separate homogeneous substances all at temperature T and pressure p, and let the products be brought finally to this same temperature T and pressure p.

Let μ_r be the potential of C_r in the pure state at pressure p and temperature T, as defined by Gibbs. Then $\mu_r = F_r$. Write

$$\Delta F = \sum \nu_r F_r = \sum \nu_r \mu_r \tag{4}$$

and, similarly, if ϕ_r denotes any thermodynamic function per gram-molecule, write

$$\Delta\phi = \sum \nu_r \phi_r.$$

It is well known that $-\Delta F$ denotes the excess of external work performed over that of mere expansion against p when the reaction proceeds reversibly. Further, when the reaction proceeds irreversibly,

$$\Delta H = \sum \nu_r H_r = -Q_p, \tag{5}$$

where Q_p, the 'heat of reaction at constant pressure', is the heat evolved.

4. Now consider an enclosure at temperature T and any pressure P containing a mixture of the substances C_r in dissociative equilibrium. It is not supposed that in the enclosure the *reacting* substances C_r are present in the proportions specified by the chemical equation—any particular substance may be present in excess or defect. Let μ'_r be the *partial* potential of C_r in the dissociative mixture in the enclosure, at the temperature T and pressure P. Then by Willard Gibbs's general condition of chemical equilibrium

$$\sum \nu_r \mu'_r = 0. \tag{6}$$

Combining this with (4), we have

$$\Delta F = \sum \nu_r(\mu_r - \mu'_r). \tag{7}$$

Now let the substance C_r be altered in pressure isothermally until it is in equilibrium with the contents of the enclosure across a suitable semi-permeable partition, provided this is possible. This is not always possible. For example, if C_r is ice, and the enclosure contains water and steam and the temperature is not that of the triple point, no pressure is possible at which the ice will be in equilibrium with the contents of the enclosure. We shall now suppose that the mixture in the enclosure is such that the process is always possible for the given substances C_r in their given phases. This imposes certain restrictions on the enclosures possible, in accordance with the phase-rule, but these restrictions are better examined specially for each particular case.

Let p_r be the pressure of C_r after it has been brought into equilibrium with the contents of the enclosure. For any homogeneous substance

$$\frac{\partial \mu_r}{\partial p} = \frac{\partial v_r}{\partial m_r} = \frac{1}{\rho_r},$$

where m_r is the mass in volume v_r; this follows from the definition of the μ_r's. Hence the change in potential during the isothermal change is given by

$$\mu'_r - \mu_r = \int_p^{p_r} \frac{dp}{\rho_r}. \tag{8}$$

Hence, by (7),

$$\Delta F = -\sum \nu_r \int_p^{p_r} \frac{dp}{\rho_r}, \tag{9}$$

where the lower limit of integration is the same for each C_r but the upper limits are in general different.

5. The quantity ΔF, referring solely to a reaction between given phases at p and T, is entirely independent of the enclosure introduced into the argument. It is thus independent of P and of the ultimate chemical analysis of the contents of the enclosure. Hence so is the right-hand side of (9).

6. Now consider another enclosure, at T, containing a neighbouring state of dissociative equilibrium not necessarily of the same ultimate chemical analysis as the former one. Let p_r+dp_r be the pressure to which C_r must be isothermally reduced in order to be in equilibrium with the contents of the new enclosure. Then

$$\Delta F = -\sum \nu_r \int\limits_{p}^{p_r+dp_r} \frac{dp}{\rho_r}. \tag{10}$$

Combining (9) and (10),

$$\sum \nu_r \frac{dp_r}{\rho_r} = 0. \tag{11}$$

In this differential equation, the ρ_r's involve only the properties of the pure substances at pressures p_r. The differentials dp_r depend only on the enclosure.

7. **Equilibrium constants.** Any integral of the differential equation (11) may be defined to be an equilibrium constant. The most useful way of constructing such an integral appears to be as follows.

Write

$$\phi_r(p_r, T) = \int\limits_{1}^{p_r} \frac{dp}{\rho_r}. \tag{12}$$

The function ϕ_r is a function of its arguments p_r and T whose form depends solely on the properties of the pure substance C_r in its given phase. We have chosen unity as the lower limit of the integral simply for convenience. Any other lower limit may be chosen; the same formulae then hold with a change of units. Now put

$$\log K = \frac{1}{RT} \sum \nu_r \phi(p_r, T) = \frac{1}{RT} \sum \nu_r \int\limits_{1}^{p_r} \frac{dp}{\rho_r}, \tag{13}$$

where R is the gas-constant. K is thus a function simply of the state of affairs in the enclosure, but its calculation for a given enclosure

depends simply on the properties of pure substances outside the enclosure. Then by (9)

$$\Delta F = -\sum \nu_r \left[\int_1^{p_r} \frac{dp}{\rho_r} - \int_1^{p} \frac{dp}{\rho_r} \right]$$

$$= -RT \log K + \sum \nu_r \int_1^{p} \frac{dp}{\rho_r},$$

or

$$\log K = -\frac{\Delta F}{RT} + \frac{1}{RT} \sum \nu_r \int_1^{p} \frac{dp}{\rho_r}. \tag{14}$$

From this equation we learn two things. First, K is independent of p, since p plays no part in the definition of K. Hence

$$-\frac{\Delta F}{RT} + \frac{1}{RT} \sum \nu_r \int_1^{p} \frac{dp}{\rho_r} \tag{15}$$

is independent of p. Secondly, the right-hand side of (14) has nothing to do with the enclosure. Hence K is the same for all enclosures at T, subject, of course, to the previously mentioned limitations imposed by the phase-rule. We call K the equilibrium constant of the reaction. It is a function of T only.

8. Behaviour of K with temperature. By equation (3″) we have

$$\frac{\partial}{\partial T}\left(\frac{\Delta F}{T}\right) = -\frac{\Delta H}{T^2}$$

$$= +\frac{Q_p(p)}{T^2}. \tag{16}$$

In $Q_p(p)$, the first p indicates that the heat of reaction meant is that at constant pressure (it distinguishes Q_p from Q_v, the 'heat of reaction at constant volume'); the second p indicates the pressure at which Q_p is measured. Now differentiate (14) totally with regard to the temperature. On the right-hand side p is arbitrary. Let us keep it constant during the differentiation, thus differentiating the right-hand side (a function of T and p) *partially* with regard to T. Using (16) we have

$$\frac{d(\log K)}{dT} = -\frac{Q_p(p)}{RT^2} + \frac{\partial}{\partial T}\left[\frac{1}{RT} \sum \nu_r \int_1^{p} \frac{dp}{\rho_r}\right]_p. \tag{17}$$

Since the left-hand side of (17) has nothing to do with p, the right-hand side of (17), namely,

$$-\frac{Q_p(p)}{RT^2}+\frac{\partial}{\partial T}\left[\frac{1}{RT}\sum \nu_r \int\limits_1^p \frac{dp}{\rho_r}\right]_p$$

must be independent of p. Using (3′) the partial differential coefficient of the last expression with regard to p is readily verified to be zero.

Equation (17) is the general form of the van 't Hoff 'reaction isochore.' It may be written in the form

$$\frac{d(\log K)}{dT} = -\frac{Q_p(p)}{RT^2}+\frac{\partial}{\partial T}\sum \nu_r\left[\int\limits_1^p \frac{pv_r}{RT}\frac{dp}{p}\right]$$

$$= -\frac{Q_p(p)}{RT^2}+\frac{\partial}{\partial T}\sum \nu_r\left[\frac{pv_r}{RT}\log p - \int\limits_1^p \frac{\partial}{\partial p}\left(\frac{pv_r}{RT}\right)\log p\, dp\right] \tag{17′}$$

For any substance C_r for which $pv_r = RT$, the contribution from the [ ] is zero. Thus each perfect gas makes zero contribution to the summation on the right-hand side. It follows that if all the C_r's are perfect gases, $Q_p(p)$ is independent of p—a well-known result.

As an application which brings out the different meanings of the two T-differentiations in (17), let us apply (17) to the case where the enclosure contains saturated vapour in contact with a condensed phase. Here the reaction is

$$[C]_c = [C]_g,$$

where the suffixes c and g denote condensed and gaseous phases respectively. The pressure P in the enclosure is here π, the saturated vapour pressure, and each p_r is also equal to π. Hence (17) is

$$\frac{d}{dT}\left[\frac{1}{RT}\left\{\int\limits_1^\pi \frac{dp}{\rho^g}-\int\limits_1^\pi \frac{dp}{\rho^c}\right\}\right] = -\frac{Q_p(p)}{RT^2}+\frac{\partial}{\partial T}\left[\frac{1}{RT}\left\{\int\limits_1^p \frac{dp}{\rho^g}-\int\limits_1^p \frac{dp}{\rho^c}\right\}\right]_p$$

Write $$\frac{1}{RT}\int\limits_1^p\left[\frac{1}{\rho^g}-\frac{1}{\rho^c}\right]dp = f(p,T).$$

Then $$\frac{d}{dT}[f(\pi,T)] = -\frac{Q_p(p)}{RT^2}+\frac{\partial}{\partial T}f(p,T).$$

Now π is a function of T. Effecting the differentiations, we have

$$f_T(\pi, T)+f_p(\pi, T)\frac{d\pi}{dT} = -\frac{Q_p(p)}{RT^2}+f_T(p, T).$$

The pressure p is arbitrary. Choose it to be π. Then

$$f_p(\pi, T)\frac{d\pi}{dT} = -\frac{Q_p(\pi)}{RT^2}$$

or

$$\left(\frac{1}{\rho^g}-\frac{1}{\rho^c}\right)_\pi \frac{d\pi}{dT} = -\frac{Q_p(\pi)}{T}.$$

This is the well-known Clapeyron equation for the change of vapour pressure with temperature. The example illustrates the flexibility permitted by the choice of p.

9. Integrated form of the reaction-isochore. In treatises on physical chemistry it is usual to proceed to integrate (17). This is in general awkward, since $Q_p(p)$ is a function of T. It is better to transform the already integrated formula (14). We have

$$\frac{\Delta F}{RT} = \frac{\Delta H - T\Delta S}{RT} = -\frac{Q_p(p)}{RT} - \frac{\Delta S}{R}.$$

Hence (14) may be written

$$\log K = \frac{Q_p(p)}{RT} + \frac{1}{RT}\sum \nu_r \int_1^p \frac{dp}{\rho_r} + \frac{\Delta S}{R}. \tag{18}$$

We now obtain expressions for $Q_p(p)$ and ΔS in terms of specific heats. By (3‴) we have

$$\frac{\partial Q_p}{\partial T} = -\Delta c_p = -\sum \nu_r c_p^r.$$

Hence

$$Q_p(p) = [Q_p(p)]_{T=0} - \sum \nu_r \int_0^T c_p^r\, dT. \tag{19}$$

Further, by (3′),

$$\frac{\partial Q_p}{\partial p} = -\sum \nu_r v_r + T\sum \nu_r \frac{\partial v_r}{\partial T},$$

whence, at $T = 0$,

$$[Q_p(p)]_{T=0} = [Q_p]_{T=0,\, p=1} - \sum \nu_r \int_1^p \frac{dp}{[\rho_r]_{T=0}}. \tag{20}$$

Hence, by (19) and (20),

$$Q_p(p) = [Q_p]_{T=0,\, p=1} - \sum \nu_r \int_0^T c_p^r\, dT - \sum \nu_r \int_1^p \frac{dp}{[\rho_r]_{T=0}}. \tag{21}$$

Any perfect gas, for which $1/\rho_r = RT/p$, makes zero contribution to the last summation, since $T = 0$. It may be assumed that the same is true for *any* gas-phase at the absolute zero. Hence the last summation need be extended to condensed phases only.

To calculate ΔS we have to evaluate S_r separately for condensed phases and gas-phases. For any condensed phase, for which c_p may be taken to vanish at $T = 0$, we may write

$$S_r = \int_0^T \frac{c_p^r}{T}\,dT + [S_0^r(p)]_{T=0}.$$

By (3), $$[S_0^r(p)]_{T=0} = [S_0^r]_{T=0,\,p=1} - \int_1^p dp\left[\frac{\partial v_r}{\partial T}\right]_{T=0}.$$

We put $$[S_0^r]_{T=0,\,p=1} = S_0^r,$$

the entropy of the gram-molecule at $T = 0$, $p = 1$. Thus

$$S_r = \int_0^T \frac{c_p^r}{T}\,dT - \int_1^p dp\left[\frac{\partial v_r}{\partial T}\right]_{T=0} + S_0^r. \tag{22}$$

For a gas-phase, for which c_p does not vanish as $T \to 0$, we use the equation (omitting the suffix r temporarily)

$$dS = \frac{c_p}{T}\,dT - \frac{\partial v}{\partial T}\,dp.$$

Then

$$S(T,p) - S(T_1,p) = \int_{T_1}^T \frac{c_p}{T}\,dT$$

$$= c_p \log T - [c_p \log T]_{T=T_1} - \int_{T_1}^T \frac{\partial c_p}{\partial T}\log T\,dT$$

$$= c_p \log T - \int_0^T \frac{\partial c_p}{\partial T}\log T\,dT - [c_p \log T]_{T=T_1} + \int_0^{T_1} \frac{\partial c_p}{\partial T}\,dT.$$

Since T_1 is arbitrary, this may be written

$$S(T,p) = c_p \log T - \int_0^T \frac{\partial c_p}{\partial T}\log T\,dT + f(p).$$

Differentiating this partially with regard to p, we have

$$-\frac{\partial v}{\partial T} = \frac{\partial c_p}{\partial p}\log T - \int_0^T \frac{\partial^2 c_p}{\partial T \partial p}\log T\,dT + f'(p),$$

or
$$f'(p) = -\frac{\partial v}{\partial T} - \int_0^T \frac{\partial c_p}{\partial p}\frac{dT}{T}$$

on integrating by parts. By (3^{iv}) this gives

$$f'(p) = \frac{\partial v}{\partial T} + \int_0^T \frac{\partial^2 v}{\partial T^2}\,dT$$
$$= -\left[\frac{\partial v}{\partial T}\right]_{T=0}.$$

Hence
$$f(p) = -\int_0^p \frac{dp}{p}\left[p\frac{\partial v}{\partial T}\right]_{T=0} + \text{const.}$$

$$= -\log p\left[p\frac{\partial v}{\partial T}\right]_{T=0} + \int_1^p \log p\,dp\left[\frac{\partial}{\partial p}\left(p\frac{\partial v}{\partial T}\right)\right]_{T=0} + S_0, \quad (23)$$

say. Finally, restoring the suffix r,

$$S_r = c_p^r \log T - \log p\left[p\frac{\partial v_r}{\partial T}\right]_{T=0} - \int_0^T \frac{\partial c_p}{\partial T}\log T\,dT +$$
$$+ \int_1^p \log p\,dp\left[\frac{\partial}{\partial p}\left(p\frac{\partial v_r}{\partial T}\right)\right]_{T=0} + S_0^r. \quad (24)$$

For a perfect gas, we have

$$p\frac{\partial v_r}{\partial T} = p\frac{\partial}{\partial T}\left(\frac{RT}{p}\right) = R, \qquad \frac{\partial}{\partial p}\left(p\frac{\partial v_r}{\partial T}\right) = 0. \quad (25),\ (26)$$

We may assume that the same hold for any gas-phase at the absolute zero. Hence (24) reduces to

$$S_r = c_p^r \log T - \int_0^T \frac{\partial(c_p^r)}{\partial T}\log T\,dT - R\log p + S_0^r. \quad (27)$$

This defines an entropy constant S_0^r for imperfect gases, the only assumption involved being that relations (25), (26) hold at $T = 0$. We have not assumed the gas perfect at higher temperatures. In (22) and (27) c_p^r is to be measured at the pressure p. Writing $\Delta S = \sum \nu_r S_r$

in (18) and using (21), (24), and (27), we have now

$$\log K = \frac{[Q_p]_{T=0,\,p=1}}{RT} + \frac{1}{R}\sum \nu_r\Bigg[-\int_1^p \frac{dp}{[\rho_r]_{T=0}} + \frac{1}{T}\int_1^p \frac{dp}{\rho_r} -$$

$$-\frac{1}{T}\int_0^T c_p^r\,dT + \int_0^T \frac{c_p^r}{T}\,dT - \int_1^p dp\left[\frac{\partial v_r}{\partial T}\right]_{T=0} + S_0^r\Bigg]_c +$$

$$+\frac{1}{R}\sum \nu_r\Bigg[\frac{1}{T}\int_1^p \frac{dp}{\rho_r} + c_p^r \log T - R\log p -$$

$$-\frac{1}{T}\int_0^T c_p^r\,dT - \int_0^T \frac{\partial c_p^r}{\partial T}\log T\,dT + S_0^r\Bigg]_g \qquad (28)$$

where the suffixes c and g denote condensed phases and gas phases respectively.

The gas-term in (28) may now be simplified. We have

$$-\frac{1}{T}\int_0^T c_p^r\,dT = -c_p^r + \frac{1}{T}\int_0^T \frac{\partial c_p^r}{\partial T}T\,dT$$

$$\frac{1}{T}\int_1^p \frac{dp}{\rho_r} = \frac{pv_r}{T}\log p - \int_1^p \frac{\partial}{\partial p}\left(\frac{pv_r}{T}\right)\log p\,dp.$$

Thus (28) becomes finally

$$\log K = \frac{[Q_p]_{T=0,\,p=1}}{RT} + \frac{1}{R}\sum \nu_r\Bigg[-\frac{1}{T}\int_1^p \frac{dp}{[\rho_r]_{T=0}} + \frac{1}{T}\int_1^p \frac{dp}{\rho_r} -$$

$$-\frac{1}{T}\int_0^T c_p^r\,dT + \int_0^T \frac{c_p^r}{T}\,dT - \int_1^p dp\left[\frac{\partial v_r}{\partial T}\right]_{T=0} + S_0^r\Bigg]_c +$$

$$+\frac{1}{R}\sum \nu_r\Bigg[c_p^r\log T - \left(R - \frac{pv_r}{T}\right)\log p - \int_1^p \frac{\partial}{\partial p}\left(\frac{pv_r}{T}\right)\log p\,dp +$$

$$+\frac{1}{T}\int_0^T \frac{\partial c_p^r}{\partial T}\,T\,dT - \int_0^T \frac{\partial c_p^r}{\partial T}\log T\,dT + S_0^r - c_p^r\Bigg]_g. \qquad (29)$$

To calculate the gas-term we require to know the behaviour of c_p^r at pressure p at all temperatures from 0 to T, and the behaviour of pv_r/T at temperature T for pressures from 1 to p, i.e. a limited

knowledge of the departures from the perfect-gas laws c_p = constant and pv_r/T = constant. For a perfect gas the integrals in the gas-term in (29) disappear, and so does the term in $\log p$. The gas-term is then independent of p, and hence the remaining terms on the right-hand side of (29) must be then independent of p. It may be readily verified that each condensed phase contributes to the summation in (29) a term independent of p.

Equation (29) is the general form of the integrated reaction isochore for any reaction in terms of measurable quantities as far as the first two laws of thermodynamics carry us.

10. Value of the equilibrium constant K. For a gas-phase, p_r is the pressure to which C_r must be reduced in order to be in equilibrium with the corresponding partial pressure in the enclosure. For an ideal gas-mixture in the enclosure, p_r is equal to the partial pressure in the gas-mixture. In general we have for the gas-contributions to $\log K$

$$\frac{1}{RT}\sum \nu_r \int\limits_1^{p_r} \frac{dp}{\rho_r} = \sum \nu_r \left[\frac{p_r v_r}{RT}\log p_r - \int\limits_1^{p_r} \frac{\partial}{\partial p}\left(\frac{pv_r}{RT}\right)\log p\, dp\right]_g,$$

which for perfect gases reduces to

$$\sum \nu_r \log p_r. \tag{30}$$

For a solid phase the value of p_r is simply P, the pressure in the enclosure, and thus the contribution from solids is

$$\sum \nu_r \left[\int\limits_1^{P} \frac{dp}{\rho_r}\right]_s. \tag{30$'$}$$

For a liquid phase, if the enclosure contains a specimen of the pure liquid the pressure p_r is again P, and we have a similar expression. If the liquid in the enclosure instead of being pure contains substances in solution of total osmotic pressure ψ_r, we have

$$\psi_r = P - p_r,$$

or

$$p_r = P - \psi_r,$$

and the contribution is

$$\sum \nu_r \left[\int\limits_1^{P-\psi_r} \frac{dp}{\rho_r}\right]_l. \tag{30$''$}$$

Equation (29) will be found to include a great variety of particular

cases usually deduced separately. In any application the aim is to choose the arbitrary value of p occurring on the right-hand side so as to obtain as simple and useful a result as possible. We now investigate particular cases.

For brevity it is useful to put

$$\Gamma_r(T,p) = -\frac{1}{T}\int_0^T c_p^r\,dT + \int_0^T \frac{c_p^r}{T}\,dT = \int_0^T \frac{dT}{T^2}\int_0^T c_p^r\,dT.$$

11. Particular cases.

CASE I. *The reacting substances all perfect gases.* We have

$$\sum \nu_r \log p_r = \frac{[Q_p]_{T=0}}{RT} + \sum \frac{\nu_r c_p^r}{R}\log T + \sum \nu_r \frac{S_0^r - c_p^r}{R}. \tag{31}$$

We ignore the suffix $p = 1$ in Q_p, since here Q_p is independent of p.

CASE II. *The enclosure includes pure specimens of the solid and liquid phases.* Choose p to be equal to P, the pressure in the enclosure. Then certain terms cancel and for perfect gases we find

$$[\sum \nu_r \log p_r]_g = \frac{[Q_p(P)]_{T=0}}{RT} + \frac{1}{R}\sum \nu_r \Gamma_r(T,P) + \frac{1}{R}\sum \nu_r\left[S_0^r - \int_1^P dp\left[\frac{\partial v_r}{\partial T}\right]_{T=0}\right]_c + \frac{1}{R}\sum \nu_r[c_p^r \log T + S_0^r - c_p^r]_g. \tag{32}$$

It will be observed that $[\sum \nu_r \log p_r]_g$ is no longer strictly independent of P; moreover, it depends on the properties of the condensed phases.

CASE III. *Only pure condensed phases in the enclosure, the reaction being between condensed phases only.* Choose p to be P. Then

$$0 = \frac{[Q_p(P)]_{T=0}}{RT} + \frac{1}{R}\sum \nu_r \Gamma_r(T,P) + \frac{1}{R}\sum \nu_r\left[S_0^r - \int_1^P dp\left[\frac{\partial v_r}{\partial T}\right]_{T=0}\right]_c. \tag{33}$$

If we take the reaction to be

$$[C]_s = [C]_l$$

where s and l denote solid and liquid phases respectively, the enclosure must be at the temperature of the melting-point at pressure P, and equation (33) accordingly determines the melting-point on putting $\nu_1 = -1$, $\nu_2 = +1$.

If we assume the Nernst Heat Theorem holds for this reaction, we have for all p

$$[\Delta S]_{T=0} = 0 \tag{34}$$

or by (22) $$-\sum \nu_r \int_1^p \left[\frac{\partial v_r}{\partial T}\right]_{T=0} dp + \sum \nu_r S_0^r = 0.$$

For this to hold for all p,

$$\sum \nu_r \left[\frac{\partial v_r}{\partial T}\right]_{T=0} = 0 \tag{35}$$

and $$[\sum \nu_r S_0^r]_c = 0. \tag{35'}$$

Hence $$0 = \frac{[Q_p(P)]_{T=0}}{RT} + \frac{1}{R}\sum \nu_r \Gamma_r(T, P). \tag{36}$$

For the Nernst Theorem to hold for a number of reactions we require that $[\partial v_r/\partial T]_{T=0}$ shall be the same for all different condensed phases of the same element, and that $[\partial v_r/\partial T]_{T=0}$ for a compound shall be the stoicheometric sum of the values for its component elements.

Planck (*Thermodynamik*, 5te Auflage (1917), S. 273) arrived at the conclusion that $(\partial v_r/\partial T)_{T=0}$ must be zero. This is more than is warranted by the Heat Theorem in the form we have quoted it, which form alone is capable of experimental test. Planck omitted the term

$$-\int_1^p \left[\frac{\partial v_r}{\partial T}\right]_{T=0} dp$$

see (22) in calculating the entropy of a condensed phase, and thus tacitly put it equal to zero before applying the Heat Theorem. Our conclusion is that $[\partial v_r/\partial T]_{T=0}$ is unrestricted for any particular element, but that the values for compounds can be formed additively.

Case IV. *Saturated vapour in the form of perfect gas in equilibrium with the pure condensed phase in the enclosure.* On the left-hand side, in $\log K$, we have $p_r = P = \pi_0$, the saturated vapour pressure at T of the pure liquid. On the right-hand side choose p to be π_0. We find

$$\log \pi_0 = \frac{[Q_p]_{T=0,\, p=\pi_0}}{RT} - \frac{1}{R}[\Gamma_r(T, \pi_0)]_c - \frac{1}{R}\left[-\int_1^{\pi_0} dp \left[\frac{\partial v_r}{\partial T}\right]_{T=0} + S_0\right]_c + \\ + \frac{1}{R}[c_p \log T + S_0^r - c_p^r]_g. \tag{37}$$

This is the well-known vapour-pressure equation in its integrated form. Put

$$I_r = \frac{[S_0^r - c_p^r]_g}{R} - \frac{[S_0^r]_c}{R}. \tag{38}$$

I_r is known as the chemical constant. The constant occurring in (31) is then given by

$$\sum \nu_r \frac{[S_0^r - c_p^r]_g}{R} = \sum \nu_r I_r - \frac{[\sum \nu_r S_0^r]_c}{R}. \tag{39}$$

Assuming the Heat Theorem to hold for this particular reaction when occurring between condensed phases, we have by (35′)

$$\sum \nu_r \frac{[S_0^r - c_p^r]_g}{R} = \sum \nu_r I_r. \tag{40}$$

This method of derivation puts in evidence the fact that to derive the known result (40) we do not need to put the separate entropy constants $[S_0^r]_c$ equal to zero. They may be chosen arbitrarily subject to (35), i.e. arbitrarily for each element.

Case V. *Vapour pressure of a solution.* Let the enclosure contain the solution of an involatile solute in contact with the vapour of the solvent, supposed in the form of perfect gas. Choose p to be π_0, the vapour pressure of the *pure* solvent. Let π be the vapour pressure of the solution, ψ the osmotic pressure of the dissolved substances. Then (39) gives

$$\log \pi - \frac{1}{RT} \int\limits_1^{\pi-\psi} \frac{dp}{[\rho_r]_c} = \frac{[Q_p(\pi_0)]_{T=0}}{RT} - \frac{1}{R}\left[\frac{1}{T}\int\limits_1^{\pi_0} \frac{dp}{\rho_r} + \Gamma_r(T, \pi_0) - \right.$$
$$\left. - \int\limits_1^{\pi_0} dp \left[\frac{\partial v_r}{\partial T}\right]_{T=0} + S_0^r\right]_c + \frac{1}{R}[c_p \log T + S_0^r - c_p^r]_g. \tag{41}$$

Comparing with (37), we have

$$\log \pi - \log \pi_0 = -\frac{1}{RT}\left[\int\limits_{\pi-\psi}^{\pi_0} \frac{dp}{\rho_r}\right]_c. \tag{42}$$

This gives the reduction in vapour pressure due to the dissolved substances. Since ρ_r is very insensitive to the pressure, it may be written approximately

$$\log \frac{\pi}{\pi_0} = -\frac{\pi_0 - (\pi - \psi)}{RT} \frac{1}{\rho_l},$$

where ρ_l is the density of the pure solvent, or again since π differs only little from π_0,

$$\frac{\pi_0-\pi}{\pi_0} = \frac{\psi+(\pi_0-\pi)}{RT}\frac{1}{\rho_l},$$

or

$$\pi_0-\pi = \pi_0\frac{\psi}{RT\rho_l-\pi_0}. \tag{43}$$

This formula is usually quoted without the small term $-\pi_0$ in the denominator. It must be remembered that ψ is the osmotic pressure when the solution is at total pressure π, the pure solvent, if in equilibrium with it across a semi-permeable membrane, then being at pressure $\pi-\psi$. The density ρ_l is to be measured in gram-molecules per cm.³

CASE VI. *Effect of external pressure on vapour pressure.* Let the enclosure, at pressure P, contain pure liquid in contact with a mixture of its vapour and an inert gas. Let π be the vapour pressure. Choose p to be the normal vapour pressure π_0. Then the general formula (29) gives

$$\log\pi-\frac{1}{RT}\int\limits_1^P\frac{dp}{[\rho]_c} = \frac{[Q_p(\pi_0)]_{T=0}}{RT}-\frac{1}{R}\Bigg[\frac{1}{T}\int\limits_0^{\pi_c}\frac{dp}{\rho_r}+\Gamma_r(T,\pi_0)-$$

$$-\int\limits_1^{\pi_0}dp\left[\frac{\partial v_r}{\partial T}\right]_{T=0}+S_0^r\Bigg]_c+\frac{1}{R}[c_p\log T+S_0^r-c_p^r]_g. \tag{44}$$

Comparing with (37)

$$\log\pi-\log\pi_0 = \frac{1}{RT}\int\limits_{\pi_0}^{P}\frac{dp}{\rho_l}$$

or approximately

$$\frac{\pi-\pi_0}{\pi} = \frac{P-\pi_0}{RT\rho_l} \tag{44'}$$

which gives the increase in vapour pressure, $\pi-\pi_0$, in terms of the excess of total pressure over normal vapour pressure, $P-\pi_0$.

This result may be deduced more simply direct from the definition of the equilibrium constant K. For K, at given T, is independent of the total pressure P in the enclosure, i.e.

$$\int\limits_1^{\pi}\frac{dp}{\rho_g}-\int\limits_1^{P}\frac{dp}{\rho_l}$$

is independent of P. Hence

$$\frac{d\pi}{\rho_g} = \frac{dP}{\rho_l}, \tag{44''}$$

which is a differential relation of which (44′) is an approximate integral.

12. Conclusion. The principal result of the present investigation is that equation (29), the general form of the integrated reaction isochore (which does not seem to have been previously given), contains a great many apparently diverse thermodynamic results as particular cases. It relates the state in all enclosures at temperature T to the properties of the pure substances occurring in the enclosure, when isolated, at an arbitrary pressure p. A by-product of the investigation is a modification of a conclusion of Planck's relating to the temperature-coefficients of expansion of condensed phases at the absolute zero.

THE UNRESTRICTED PLANE PARTITION

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I HAVE recently discussed a method of determining the generating functions of various types of plane partition,* among which MacMahon's formula

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}... \qquad (1)$$

for the generating function of the unrestricted plane partition† is included as a special case, but this method can scarcely be regarded as furnishing an 'intuitive' proof of (1) comparable with the 'intuitive' argument for Euler's formula

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}...$$

for the generating function of the unrestricted linear partition. I therefore attempt here the fabrication of an *ad hoc* intuitive proof.

In the first place it is intuitive that (1) is the generating function of the partition formed of a *set* of linear partitions in which 1 may be used as a part in only one linear partition, 2 in only two such partitions, 3 in only three, and so on. We must adopt some convention to keep the several linear partitions distinct, since interchange of any two of them changes the partition as a whole. Although there is, as yet, no obligation to think geometrically, we may conveniently distinguish the linear partitions by arranging them in the following order: first the linear partition in which the parts 1, 2, 3, 4,... are all admissible; then the partition admitting only parts 2, 3, 4,...; then the partition admitting only parts 3, 4,..., and so on.

Now denote by $\{\pi_k\}$ the aggregate of all sets of k linear partitions, unrestricted alike in number of parts, magnitude of parts, and number partitioned. Thus $\{\pi_1\}$ means the aggregate of all unrestricted linear partitions. In each partition of $\{\pi_1\}$ insert the same additional part a: that is to say, the a-group in each such partition is to be increased by one extra term. Denote the aggregate of the partitions so obtained by $\{\pi_1\,|a\}$: clearly they are all to be found in $\{\pi_1\}$ itself. Denote by $\{\pi_1\}-\{\pi_1\,|a\}$ the aggregate of those partitions found in $\{\pi_1\}$ but not in $\{\pi_1\,|a\}$. It evidently comprises just those partitions of $\{\pi_1\}$ in which a does not appear as a part.

* *Quart. J. of Math.* (Oxford) 2 (1931), 234–40.

† *Combinatory Analysis* (1916), vol. ii, p. 243, § 495, the last formula.

We may extend this notation to denote by $\{\pi_1\,|a,b\}$ the aggregate of partitions obtained from $\{\pi_1\}$ by inserting two additional parts a, b; and to denote by $\{\pi_2\,|a|b\}$ the aggregate obtained from $\{\pi_2\}$ by inserting an additional part a in the first of its two linear partitions and an additional part b in the second of them. Thus

$$\{\pi_1\}-\{\pi_1\,|a\}-\{\pi_1\,|b\}+\{\pi_1\,|a,b\} \qquad (a \neq b)$$

denotes the aggregate of unrestricted linear partitions in which neither a nor b is used as a part;

$$\{\pi_2\}-\{\pi_2\,|a|0\}-\{\pi_2\,|0|b\}+\{\pi_2\,|a|b\}$$

denotes the aggregate of pairs of unrestricted linear partitions in which a does not occur in the first partition, nor b in the second partition.

Further generalizations of the notation with analogous implications become obvious, and we see that the aggregate of those sets of exactly k linear partitions which are generated by (1) may be written in the form

$$\sum (-)^s\{\pi_k\,|0|a_{21}|a_{31},a_{32}|...|a_{k1},...,a_{k,\,k-1}\}, \tag{2}$$

where the symbols a_{r1}, a_{r2},..., $a_{r,\,r-1}$ in the rth cell $|\ |$ are either the $r-1$ numbers 1, 2,..., $r-1$ or else a selection of them, without repetitions, eked out by zeros; and s is the total number of such symbols a which are not zero.

Now those aggregates of partitions in (2) in which the sums $a_{21}+a_{31}+...+a_{k,\,k-1}$ are equal give equally numerous partitions of equal numbers, and, since the purpose of a generating function is to count, not to describe, partitions, it is only such sums in (2) that we need consider. Take $k = 5$ for brevity. Then evidently these sums $a_{21}+a_{31}+...+a_{k,\,k-1}$ are the various increments obtained in the sum of the first line of the scheme

$$\begin{array}{cccccccccc} 0+&1+&0+&2+&1+&0+&3+&2+&1+&0 \\ (1&2&2&3&3&3&4&4&4&4), \end{array} \tag{3}$$

when some or all of the figures in that line are replaced by the figures immediately beneath, s being the number of such replacements.

Now, throughout the scheme (3), let us interchange a specified pair of numbers, say 0 and 4. We get

$$\begin{array}{cccccccccc} 4+&1+&4+&2+&1+&4+&3+&2+&1+&4 \\ (1&2&2&3&3&3&0&0&0&0). \end{array}$$

Since the transference of complete columns does not alter the total

sum, let us interchange the first, third, and sixth columns with the ninth, eighth, and seventh columns respectively. We get

$$\begin{array}{cccccccccc} 1+ & 1+ & 2+ & 2+ & 1+ & 3+ & 4+ & 4+ & 4+ & 4 \\ (0 & 2 & 0 & 3 & 3 & 0 & 3 & 2 & 1 & 0), \end{array}$$

which may be obtained from (3) by vertical interchanges in the seven columns

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 4 & 1 & 2 & 3 & 4 & 4 & 4. \end{array}$$

So, in general, the complete interchange throughout (3) of a pair of numbers r, s can be effected, apart from transference of complete columns, by interchanging r, s vertically, and then each of them with all other numbers. The complete interchange is therefore accomplished in an *odd* number of vertical interchanges, and so changes the sign attached to the partition. Thus those sums in (3) in which two figures occur equally often cancel in the original summations $\sum$ of (2). Now there are in all $\frac{1}{2}k(k-1)$ terms in (3) and its permutations, and no one number may occur more than $k-1$ times. Thus, since no two numbers may occur equally often, only those sets survive the summation in which just one number occurs once, just one number occurs twice,..., just one number occurs $k-1$ times.

If we collect like numbers and indicate frequency of appearance by an index, then the original arrangement of the upper line in (3) may be indicated by

$$0^4 1^3 2^2 3^1 4^0,$$

and the only effective variations are those obtained by permuting the suffixes, the sign in $\sum$ changing at each permutation.

I now correlate these permutations with permutations of the set of k linear partitions, by building up the latter into two-dimensional partitions. These 'plane partitions' will not yet pass the test of MacMahon's definition: the columns will not necessarily be arranged monotonically, nor will the left-hand edge of the plane partition be always rectilinear and vertical. But I go on to show that they can be correlated in pairs of opposite sign so as to leave surviving just the plane partitions of MacMahon. I construct my two-dimensional partitions thus. In each linear partition arrange the parts from left to right in monotonic non-ascending order, and in the standard partition corresponding to $0^4 1^3 2^2 3^1 4^0$ itself, arrange the linear partitions one over the other, so that they start on the left in the same vertical.

Into such a standard partition represented diagrammatically thus

$$\begin{array}{ccccc} \times & \times & \times & \times & \cdots \\ \times & \times & \times & \times & \cdots \\ \times & \times & \times & \times & \cdots \\ \times & \times & \times & \times & \cdots \\ \times & \times & \times & \times & \cdots \end{array} \tag{4}$$

I have inserted a 'staircase'. In a pair of consecutive rows of this partition I interchange the sets of terms to the left of the 'staircase', decreasing or increasing each term by unity according as it ascends or descends. In $0^4 1^3 2^2 1^3 4^0$ we may regard the base figure as indicating the terms in the several rows to the left of the staircase and the indices as measuring the relative importance of the rows. We then see that an 'escalade', if I may be allowed the term, diminishes the number partitioned by exactly the increment obtained in $0^4 1^3 2^2 3^1 4^0$ on interchanging the corresponding figures. To restore the partitioned number we must add this increment, which we have seen to be $a_{21}+a_{31}+...+a_{k,k-1}$ of (2). Thus, if we change the sign of the partition at every 'escalade', the aggregate of the partitions obtained by one or many 'escalades' has one-to-one correspondence with the aggregate (2) and therefore has (1) as its generating function. I remark also that, wherever, in the partition, the 'staircase' be drawn, the 'escalade' diminishes the partitioned number by the same amount.

I come now to the monotonic ordering of the columns, the rows being already in non-ascending order. Suppose (i) that in one of the partitions occurs a broken column a_s, b_s, c_s-c_{s+1}, d_{s+1}, e_{s+1}, in which successive terms increase or decrease as shown in diagram (5) below, (ii) that *everywhere to the right* of this broken column the straight columns descend in unbroken monotonic (non-ascending) order, and (iii) that nothing is asserted about monotonic descent to the *left* of the broken column

$$\begin{array}{ccccc} \cdots & a_{s-1} & a_s & a_{s+1} & \cdots \\ & & \geqslant & & \\ \cdots & b_{s-1} & b_s & b_{s+1} & \cdots \\ & & \wedge & \geqslant & \\ \cdots & c_{s-1} & c_s & c_{s+1} & \cdots \\ & & & \geqslant & \\ \cdots & d_{s-1} & d_s & d_{s+1} & \cdots \\ & & & \geqslant & \\ \cdots & e_{s-1} & e_s & e_{s+1} & \cdots \end{array} \tag{5}$$

The inequality $\wedge$ may occur anywhere in the column, which always breaks immediately below it, except, naturally, when it is at the foot of the column. Briefly, we choose the inequality $\wedge$ which is firstly farthest to the right and then highest. It is obvious that in a given partition there can be at most one such column.

I now perform an 'escalade' upon the two rows which are separated by the sign $\wedge$. The parts $b_1,\ldots, b_{s-1}$, each increased by unity, are brought down in undisturbed order immediately to the front of c_{s+1}; the parts $c_1,\ldots, c_s$, each diminished by unity, are raised in undisturbed order immediately to the front of b_s. It is clear that the inequalities shown or implied in (5), monotonic ordering of the rows included, are invariant for the 'escalade'. If, exceptionally, b_s had been the first part of its row, we should have had before and after the 'escalade' the respective diagrams

$$\begin{array}{cccccccccc}
\cdots & a_s & a_{s+1} & \cdots & \qquad & & \cdots & a_s & a_{s+1} & \cdots \\
 & \rotatebox[origin=c]{-90}{$\geqslant$} & & & & & & \rotatebox[origin=c]{-90}{$\geqslant$} & & \\
 & b_s & b_{s+1} & \cdots & & \cdots & c_s-1 & b_s & b_{s+1} & \cdots \\
 & \wedge & \rotatebox[origin=c]{-90}{$\geqslant$} & & & & & & \rotatebox[origin=c]{-90}{$\geqslant$} & \\
\cdots & c_s & c_{s+1} & \cdots & & & & & c_{s+1} & \cdots \\
 & & \rotatebox[origin=c]{-90}{$\geqslant$} & & & & & & \rotatebox[origin=c]{-90}{$\geqslant$} & \\
\cdots & d_s & d_{s+1} & \cdots & & & \cdots & d_s & d_{s+1} & \cdots \\
 & & \rotatebox[origin=c]{-90}{$\geqslant$} & & & & & & \rotatebox[origin=c]{-90}{$\geqslant$} & \\
\cdots & e_s & e_{s+1} & \cdots, & & & \cdots & e_s & e_{s+1} & \cdots,
\end{array}$$

so that we consider columns broken not only by a $\wedge$ but also by a gap in the partition. In this wider sense, every partition of our aggregate (with one class of exception) has one and only one such broken column to the right of which there is everywhere monotonic descent; and partitions with similar broken columns similarly placed can be paired off by 'escalades'.

All such partitions in our aggregate therefore cancel out in pairs and there remain only the partitions of the exceptional class having no broken column. These residual partitions, which therefore have (1) as their generating function, are evidently the partitions in which there are no gaps in the left-hand edge and where there is everywhere monotonic descent. Clearly these are the unrestricted plane partitions of MacMahon, and we have therefore completed our 'intuitive' proof that (1) is the generating function of the unrestricted plane partition.